Studies in the Physical Foundations of Gravitational Theories

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Overview

- Special relativity: simultaneity, length measurement, accelerated observers
- Hypothesis of Locality and its problems
- Global concepts, Radiation
- Electrodynamics and accelerated observers
- Equivalence principle and gauge theories of gravity
- Teleparallel Theories

Main topics

1. Problems with traditional treatment of accelerated observers, using the Hypothesis of Locality: length measurements, linear and rotational acceleration
2. Radiation of a uniformly accelerated charge
3. Alternative nonlocal approaches in electrodynamics
4. Alternative to traditional application of Principle of Equivalence: teleparallel theories

SR: Simultaneity and Lengths

Event: single location/position (coordinate label on an indefinitely extended rigid ruler), single instant in time.

Inertial observers can use synchronized clocks: They can correct for travel time of a signal. Prior knowledge needed: Distance between source of signal and observer.

Length in a reference frame := difference between coordinate positions at the same time (*def. with rulers*).

Different inertial frames (with relative velocity):
Lorentz-Fitzgerald contraction: $l' = \sqrt{1 - \frac{v^2}{c^2}}\, l = \frac{1}{\gamma} l_0$.

Alternative, operational length definition: Observer 1 sends signal to observer 2 who sends signal immediately back.
$L := \frac{1}{2} c \Delta t$ (Assumption: c constant in all reference frames).

Acceleration lengths

Orthonormal frame field $\lambda^{\mu}{}_{(\alpha)}(\tau)$. Covariant derivative of the frame field:

$$\frac{D\lambda^{\mu}{}_{(\alpha)}}{D\tau} = \Phi_{\alpha}{}^{\beta}(\tau)\lambda^{\mu}{}_{(\beta)} \; .$$

For vanishing non-metricity (using orthonormality): $\Phi_{\alpha\beta}(\tau) = -\Phi_{\beta\alpha}(\tau)$. Therefore:

$$\Phi_{\alpha\beta} := \left[\begin{array}{c|c} 0 & \vec{a}/c \\ \hline -\vec{a}/c & \vec{\Omega} \end{array}\right]$$

Scalar invariants of antisymmetric tensor $\Phi_{\alpha\beta}$:

$$I = \frac{1}{2c^2}\Phi_{\alpha\beta}\Phi^{\alpha\beta} = -\frac{a^2}{c^4} + \frac{\Omega^2}{c^2} \,,$$

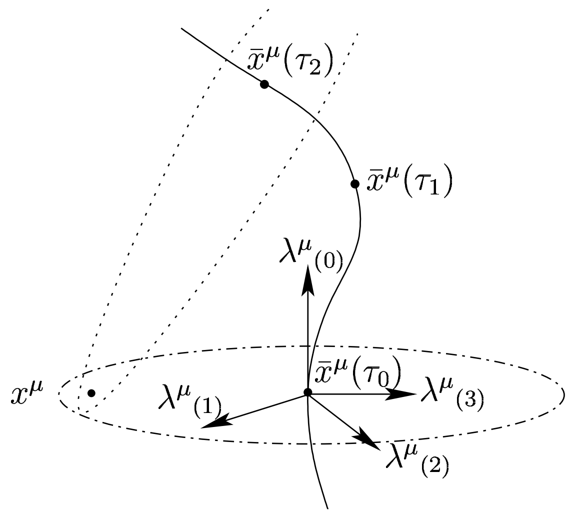
$$I^* = \frac{1}{4c^2}\Phi^*_{\alpha\beta}\Phi^{\alpha\beta} = \frac{\vec{a}}{c^2} \cdot \frac{\vec{\Omega}}{c} \,.$$

Proper acceleration lengths $\mathcal{L}$: $\frac{c^2}{a}$ and $\frac{c}{\Omega}$.

Earth surface:

$$\frac{c^2}{a} = \frac{(3 \cdot 10^8 \,\frac{\mathsf{m}}{\mathsf{s}})^2}{9.8 \frac{\mathsf{m}}{\mathsf{s}^2}} = 9.46 \cdot 10^{15}\,\mathsf{m} \approx 1\,\mathsf{ly}$$

$$\frac{c}{\Omega} = \frac{3 \cdot 10^8 \,\frac{\mathsf{m}}{\mathsf{s}}}{7.272 \cdot 10^{-5}\,\mathsf{s}^{-1}} = 4.1253 \cdot 10^{12}\,\mathsf{m} \approx 27.5\,\mathsf{AU}$$



New accelerated, geodesic coordinates using only position and basis frame field: $X^\mu = (c\tau, \vec{X})$

$$x^\mu(\tau) = \bar{x}^\mu(\tau) + X^i \lambda^\mu{}_{(i)}(\tau)$$

Metric for accelerated observer in Minkowski spacetime:

$$ds^2 = o_{\mu\nu}\, dx^\mu\, dx^\nu = \left[\left(1 + \frac{\vec{a}\cdot\vec{X}}{c^2}\right)^2 - \left(\frac{\vec{\Omega}\times\vec{X}}{c}\right)^2\right](dx^0)^2 - 2\left(\frac{\vec{\Omega}\times\vec{X}}{c}\right)\cdot d\vec{X} dx^0 - \delta_{ij}\, dX^i\, dX^j \;.$$

coordinates admissible as long as $\left(1 + \frac{\vec{a}\cdot\vec{X}}{c^2}\right)^2 > \frac{1}{c^2}\left(\vec{\Omega}\times\vec{X}\right)^2$.

Hypothesis of Locality

> An accelerated observer measures the same physical results as a standard inertial observer that has the same position and velocity at the time of measurement.

Clock hypothesis: Restricted hypothesis of locality for time measurements only.

Hypothesis ingrained in *Newton's theory*, a theory for point particles:

All forces and movements are determined by a second order equation of motion. It determines the state of a particle $(\vec{x}, \vec{v})$ once the initial condition is specified.

- **Waves**: $\omega' = \gamma\left(\omega - \vec{v}\cdot\vec{k}\right)$.

  For accelerated observers: v changes.

  Measurement of frequency only possible, if velocity doesn't change too much over a period of the wave: $T\left|\frac{d\vec{v}}{dt}\right| \ll v$.

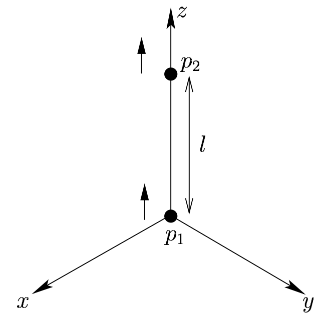
  With $\lambda = cT$, we get $\frac{\lambda}{c}a \ll v < c$, and thus $\lambda \ll \frac{c^2}{a}$.

- **Charged particles**: Accelerated particles radiate. Described by Abraham-Lorentz-Dirac equation

$$m\frac{d^2\vec{x}}{dt^2} - \frac{2}{3}\frac{q^2}{c^3}\frac{d^3\vec{x}}{dt^3} + \cdots = \vec{F}$$

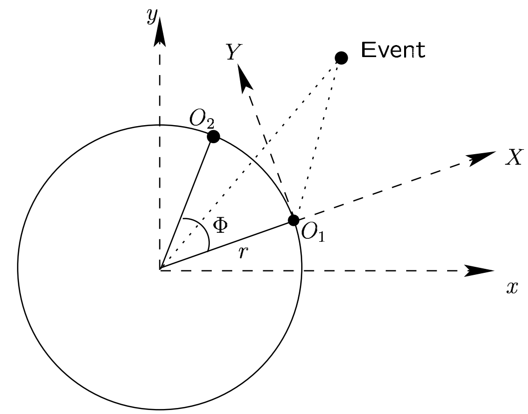
- **Quantum mechanical particles**: have Compton and de Broglie wavelengths associated with them.

Linear acceleration: Two observers, a distance l apart (in initial inertial frame), with identical acceleration profiles



- Distance in momentarily comoving inertial frame: $l' = \frac{1}{\sqrt{1-\frac{v^2(t)}{c^2}}}\, l = \gamma(t) l$
- Distance in accelerating frame with P_1 in origin: $\frac{L}{l'} = 1 - \frac{1}{2}\beta^2\gamma\epsilon + \frac{1}{2}\beta^2\gamma^2\epsilon^2 + \mathcal{O}(\epsilon^3)$ with $\epsilon = \frac{l}{\frac{c^2}{g}}$.
- Distance in accelerating frame with P_2 in origin: $\frac{L'}{l'} = 1 + \frac{1}{2}\beta^2\gamma\epsilon + \frac{1}{2}\beta^2\gamma^2\epsilon^2 + \mathcal{O}(\epsilon^3)$.
- Distance according to operational definition for P_1: $\frac{L^*}{l'} = 1 - \frac{1}{2}\gamma\epsilon(1+\beta^2) + \mathcal{O}(\epsilon^2)$.
- Distance according to operational definition for P_2: $\frac{L'^*}{l'} = 1 - \frac{1}{2}\gamma\epsilon(1-\beta^2) + \mathcal{O}(\epsilon^2) = 1 - \frac{1}{2}\frac{\epsilon}{\gamma} + \mathcal{O}(\epsilon^2)$.

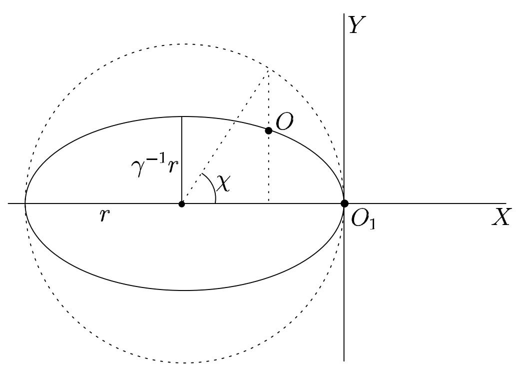
Acceleration due to Rotation: Two observers on circle with radius r with angle Φ between them.



– Distance in inertial frame: $l = r\Phi$.

– Dist. in momentarily comoving frame: $l' = \gamma l = \gamma r \Phi$.

– Distance in accelerating frame with O_1 in origin:



$$D = r \int_0^{\Delta} \sqrt{1 - \beta^2 \cos^2 \chi}\, d\chi$$

For $\beta^2 \ll 1$:

$$\frac{D}{l'} = 1 - \frac{3}{4}\beta^2 \left(1 + \frac{\sin 2\Phi - 8 \sin \Phi}{6\Phi}\right) + \mathcal{O}(\beta^4).$$

Radiation

Controversial question: Does a uniformly accelerated charge radiate?

Yes, with standard Larmor formula:

$$\mathcal{R} = \frac{2}{3}e^2 g^2 = \frac{2}{3}e^2 a_\mu a^\mu \ .$$

From classical Dirac equation of motion for a charged point particle, we calculate: *no* radiation reaction in this case. Not physically reasonable, so the Dirac equation of motion is incomplete.

No contradiction with principle of equivalence: The principle of equivalence is only *locally* valid, and radiation is a *global* concept.

Alternatives to Hypothesis of Locality in EM theory

1. **Mashhoon model**: The field that an accelerated observer actually measures depends linearly, but nonlocally on inertial measurements:

$$\mathcal{F}_{\alpha\beta}(\tau) = F_{\alpha\beta}(\tau) + \int_{\tau_0}^{\tau} K_{\alpha\beta}{}^{\gamma\delta}(\tau,\tau')\, F_{\gamma\delta}(\tau')\, d\tau' \,,$$

   Kernel K is expected to depend on the acceleration of the observer.

2. **Charge & Flux model**: The constitutive relation is linear, but nonlocal:

$$\mathcal{H}^{\alpha\beta}(\tau,\xi) = \sqrt{-g}\, g^{\alpha\mu}\, g^{\beta\nu} \int \mathcal{K}_{\mu\nu}{}^{\rho\sigma}(\tau,\tau',\xi) F_{\rho\sigma}(\tau',\xi)\, d\tau' \ ,$$

ξ depends on the medium. The kernel will be acceleration-dependent, if we use the connection of the accelerated observer:

$$\mathcal{H}^{\alpha\beta}(\tau) = \sqrt{-g}\, g^{\alpha\mu}\, g^{\beta\nu} \Big[F_{\mu\nu}(\tau) - c \int_{\tau_0}^{\tau} [\Gamma_{0\mu}{}^{\rho}(\tau-\tau') F_{\rho\nu}(\tau') + \Gamma_{0\nu}{}^{\rho}(\tau-\tau') F_{\mu\rho}(\tau')]\, d\tau' \Big] ,$$

For *constant* linear acceleration $\boldsymbol{a}$ and uniform rotation $\boldsymbol{\omega}$, the two models are the same, provided we identify $\mathcal{H}$ with $\mathcal{F}$.

This agreement does *not* extend to the case of nonuniform acceleration.

Equivalence Principle

> Observers in a gravitational field and accelerated observers in Minkowski spacetime measure the same physics *locally*.

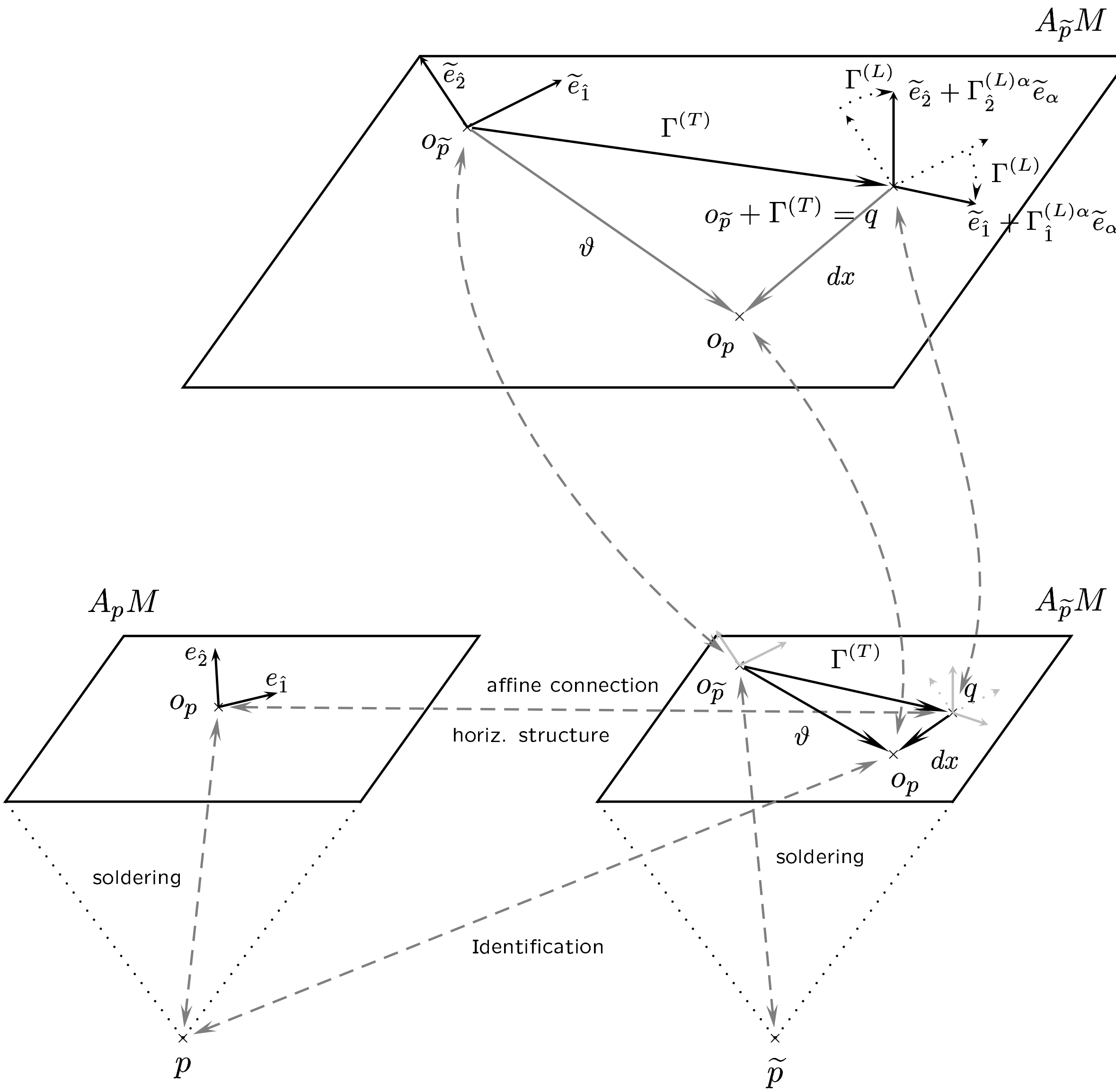
The question what accelerated observers measure arises at the core of GR.

Einstein's Principle of Equivalence *and* Hypothesis of Locality $\Rightarrow$ every observer in a gravitational field is pointwise inertial.

Spacetime manifold can then locally be substituted by a Minkowski spacetime.

Question: How to connect neighboring Minkowski spacetimes (neighboring affine tangent spaces)?

Soldering and affine connection



Metric-affine gauge theories

General Lagrangian (minimally coupled):

$$L_{\text{MAG}} = V_{\text{MAG}}(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R_\alpha{}^\beta) + L_{\text{matter}}(g_{\alpha\beta}, \vartheta^\alpha, \Psi, D\Psi) \, .$$

Potentials:

- coframe ϑ^α,
- connection $\Gamma_\alpha{}^\beta$,
- metric $g_{\alpha\beta}$.

Field strengths:

- torsion $T^\alpha = D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta$,
- non-metricity $Q_{\alpha\beta} = -Dg_{\alpha\beta}$,
- curvature $R_\alpha{}^\beta = \text{“}D\Gamma_\alpha{}^\beta\text{”} = d\Gamma_\alpha{}^\beta + \Gamma_\gamma{}^\beta \wedge \Gamma_\alpha{}^\gamma$.

Connection $\Gamma_\alpha{}^\beta$ and curvature $R_\alpha{}^\beta$ are general (not just Christoffel connection and Riemannian curvature); We allow for

- energy-momentum current (translation)
- spin current (rotation)
- dilation current (length change)
- shear current (angle change)

when a vector is transported along an infinitesimally small, closed curve.

Special cases

Vanishing non-metricity: Poincaré-Lagrangian:

$$V_{\mathsf{PG}} = \frac{1}{2\ell^2}\Big[- a_0 R^{\alpha\beta} \wedge \eta_{\alpha\beta} + T^\alpha \wedge {}^\star \left(\sum_{I=1}^{3} a_I \, {}^{(I)}T_\alpha \right) - \frac{1}{2} R^{\alpha\beta} \wedge {}^\star \left(\sum_{I=1}^{6} b_I \, {}^{(I)}R_{\alpha\beta} \right) \Big] .$$

Einstein's theory: vanishing non-metricity, vanishing torsion (via Lagrange multipliers).

Lagrangian *linear* in curvature:

$$V_{\mathsf{Einstein}} = \frac{1}{2\ell^2} {}^\star(\vartheta^\alpha \wedge \vartheta^\beta) \wedge R_{\alpha\beta} \qquad (\ell = \text{Planck's length}) .$$

Teleparallel theory: vanishing non-metricity, vanishing curvature (via Lagrange multipliers).

Lagrangians *quadratic* in torsion

$V_{||} = \frac{1}{2\ell^2}\left(\rho_1\,{}^{(1)}V + \rho_2\,{}^{(2)}V + \rho_4\,{}^{(4)}V\right)$, with:

$${}^{(1)}V = T^\alpha \wedge {}^\star T_\alpha \qquad \text{(pure Yang-Mills type)}\,,$$

$${}^{(2)}V = \left(T_\alpha \wedge \vartheta^\alpha\right) \wedge {}^\star \left(T_\beta \wedge \vartheta^\beta\right)\,,$$

$${}^{(4)}V = \left(T_\alpha \wedge \vartheta^\beta\right) \wedge {}^\star \left(T_\beta \wedge \vartheta^\alpha\right)\,.$$

This Lagrangian is *equivalent* to Einstein's theory for

$$\rho_1 = 0, \quad \rho_2 = -\frac{1}{2}, \quad \rho_4 = 1\,.$$

Special choices for parameters ρ_i

	$\text{GR}_{\parallel}$	vdH	viable	YM	YM†	KI
ρ_1	0	0	0	1	1	2
ρ_2	$-\frac{1}{2}$	0	arb.	0	0	0
ρ_4	1	1	1	0	-1	-1

- *viable* teleparallel theories: agree with first post-Newtonian approx. of GR.
- *vdH* (von der Heyde Lagrangian): most simple viable theory.
- *YM* (Yang-Mills type Lagrangian): teleparallel Lagrangian, using only the Rumpf-Lagrangian of Yang-Mills type.
- *YM*†: the exterior derivative d is substituted by the co-derivative $d^\dagger := -{}^\star d^\star$ in YM.
- *KI*: Lagrangian $\text{YM} + \text{YM}^\dagger$ (corrected version of the original suggestion by Kaniel and Itin).

Other decompositions

The teleparallel Lagrangians can be split up in *irreducible parts*:

$$V_{\|} = \frac{1}{2\ell^2}\sum_{I=1}^{3} a_I \left(D\vartheta_\alpha \wedge {}^\star\, {}^{(I)}D\vartheta^\alpha \right)$$

with

$$
\begin{aligned}
{}^{(1)}T^\alpha &= {}^{(1)}D\vartheta^\alpha := D\vartheta^\alpha - {}^{(2)}D\vartheta^\alpha - {}^{(3)}D\vartheta^\alpha && \text{(tentor)},\\
{}^{(2)}T^\alpha &= {}^{(2)}D\vartheta^\alpha := \frac{1}{3}\,\vartheta^\alpha \wedge \left(e_\beta \rfloor D\vartheta^\beta\right) && \text{(trator)},\\
{}^{(3)}T^\alpha &= {}^{(3)}D\vartheta^\alpha := -\frac{1}{3}\,{}^\star\left[\vartheta^\alpha \wedge {}^\star\left(\vartheta^\beta \wedge D\vartheta_\beta\right)\right] \\
&= \frac{1}{3}\, e_\alpha \rfloor \left(\vartheta^\beta \wedge D\vartheta_\beta\right) && \text{(axitor)}.
\end{aligned}
$$

Other decompositions are possible: for example *Møller's tetrad theory*.

Coefficients of different splittings are uniquely determined, e.g. between Rumpf-Lagrangians and irreducible decomposition.

$$\rho_1 = \frac{1}{3}\,(a_2 + 2a_1)\ , \quad \rho_2 = \frac{1}{3}\,(a_3 - a_1)\ , \quad \rho_4 = \frac{1}{3}\,(a_1 - a_2)$$

and the inverse

$$a_1 = \rho_1 + \rho_4\ , \qquad a_2 = \rho_1 - 2\rho_4\ , \qquad a_3 = \rho_1 + 3\rho_2 + \rho_4\ .$$

Calculations can be done in most convenient splitting.

PPN parameters in simple metrics

One choice of post-Newtonian parameters leads to the metric

$$g_{00} = 1 - 2U + 2\beta U^2$$
$$g_{0i} = \frac{1}{2}\,(4\gamma + 4 + \alpha)\,V_i$$
$$g_{ij} = (-1 - 2\gamma U)\,\delta_{ij}$$

with

$$U = \frac{GM}{c^2 r} \quad \text{and} \quad V_i = -\frac{G}{2}\epsilon_{ijk}\frac{x^j J^k}{c^3 r^3}\ .$$

For Einstein's theory: $\gamma = 1$, $\beta = 1$, and $\alpha = 0$.

Conversion to coframe via $g = g_{ij}\, dx^i \otimes dx^j = o_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta$.

PPN parameters in a coframe

$$
\begin{aligned}
\vartheta^0 &= a_0\, dt\ , \\
\vartheta^1 &= b_0\, dt + b_1\, dx\ , \\
\vartheta^2 &= c_0\, dt \qquad\qquad\qquad + c_2\, dy\ , \\
\vartheta^3 &= d_0\, dt \qquad\qquad\qquad\qquad\qquad + d_3\, dz\ .
\end{aligned}
$$

with

$$
\begin{aligned}
a_0 &= 1 - U + \left(\beta - \frac{1}{2}\right) U^2\ , \\
b_1 = c_2 = d_3 &= 1 + \gamma U\ , \\
b_0 = c_0 = d_0 &= \frac{1}{2}\,(4\gamma + 4 + \alpha)\, V_i\ .
\end{aligned}
$$

Spherically symmetric case

No rotation: $J^k = 0$, so $V_i = 0$. The parameter α is not in metric or coframe anymore.

- The second Rumpf-Lagrangian (ρ_2) is always fulfilled for any spherically symmetric ansatz.
- ρ_1-Lagrangian: demands $\gamma = 0$. The Yang-Mills Lagrangian does *not* lead to a viable post-Newtonian theory.
- ρ_4-Lagrangian: demands $\gamma = 1$. Compatible with Einstein's theory.
- In the calculated order, β can be arbitrary.

Axially symmetric case

J^k arbitrary. Since the ρ_1-Lagrangian is already excluded by the symmetrical case, it's not considered here.

- In the calculated order, the ρ_2-Lagrangian allows arbitrary PPN parameters.
- ρ_4-Lagrangian: demands $\gamma = 1$. Compatible with Einstein's theory.
- In the calculated order, α and β can be arbitrary.

Summary

- Hypothesis of locality: Demonstration of basic limitations on length measurement by accelerated observers; they are more severe than those imposed by requirement of admissibility of coordinates
- Global concepts: Investigation of radiation
- Alternatives to hypothesis of locality: Nonlocal electromagnetic theories
- Einstein's principle of equivalence: metric-affine gauge theories are viable; specifically determination of viable teleparallel theories