A small guide to variations in teleparallel gauge theories of gravity and the Kaniel-Itin model

Uwe Muench, Frank Gronwald, Friedrich W. Hehl*

Abstract

Recently Kaniel & Itin proposed a gravitational model with the wave type equation $[\Box + \lambda(x)]\vartheta^\alpha = 0$ as vacuum field equation, where ϑ^α denotes the coframe of spacetime. They found that the viable Yilmaz-Rosen metric is an exact solution of the tracefree part of their field equation. This model belongs to the *teleparallelism* class of gravitational gauge theories. Of decisive importance for the evaluation of the Kaniel-Itin model is the question whether the variation of the coframe commutes with the Hodge star. We find a master formula for this commutator and rectify some corresponding mistakes in the literature. Then we turn to a detailed discussion of the Kaniel-Itin model. *file kaniel21.tex, 1998-01-12*

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1 Introduction

We were very much surprised when we learned during the 8$^{\text{th}}$ Marcel Grossmann Meeting in Jerusalem [1] that Kaniel & Itin [2] were able to propose a gravitational model which looks viable at a first sight even if it had neither an Einstein-Hilbert type of Lagrangian nor the Schwarzschild metric as an exact solution. Their gravitational potential is represented by a quartet of 1-forms $\vartheta^{\hat{0}}, \vartheta^{\hat{1}}, \vartheta^{\hat{2}}, \vartheta^{\hat{3}}$ or, for short, by ϑ^α, which constitutes the coframe field of spacetime. Their vacuum field equation is simply the wave equation with an additional 'massive' contribution depending on some scalar field $\lambda(x)$:

$$[\Box + \lambda(x)]\,\vartheta^\alpha = 0\,. \tag{1}$$

*Institute for Theoretical Physics, University of Cologne, D-50923 Köln, Germany

They show that the Yilmaz-Rosen metric [3, 4] solves the *tracefree* part of (1) exactly.

Let us be a bit more specific: The Yilmaz-Rosen metric, in isotropic coordinates, is given by

$$g = e^{-\frac{2m}{r}}\, dt^2 - e^{\frac{2m}{r}}\, \left(dx^2 + dy^2 + dz^2\right)\,, \tag{2}$$

where $r^2 := x^2 + y^2 + z^2$. If we introduce an orthonormal coframe,

$$g = o_{\alpha\beta}\, \vartheta^\alpha \otimes \vartheta^\beta \qquad \text{with} \qquad o_{\alpha\beta} = \text{diag}(+1, -1, -1, -1)\,, \tag{3}$$

then the following coframe, up to arbitrary local Lorentz transformations, represents the Yilmaz-Rosen metric:

$$\vartheta^{\hat{t}} = e^{-\frac{m}{r}}\, dt\,, \quad \vartheta^{\hat{x}} = e^{\frac{m}{r}}\, dx\,, \quad \vartheta^{\hat{y}} = e^{\frac{m}{r}}\, dy\,, \quad \text{and} \quad \vartheta^{\hat{z}} = e^{\frac{m}{r}}\, dz\,. \tag{4}$$

The tracefree part of (1) will be determined in Sec. 5.2 and turns out to be

$$\left[\Box - \frac{1}{4}(e_\beta \rfloor \Box \vartheta^\beta)\right] \vartheta^\alpha = 0\,. \tag{5}$$

The coframe (4) solves the tracefree field equation (5) exactly. We have verified this by means of our computer algebra program `kaniti.exi` displayed in the appendix in Sec. B.

Kaniel & Itin tried to derive the field equation (1) from a suitable Lagrangian. For that purpose they had to assume specifically that the variation $\delta\vartheta^\alpha$ of the coframe ϑ^α commutes with the Hodge star: ${}^\star\delta\vartheta^\alpha = \delta\, {}^\star\vartheta^\alpha$. However, such a commutativity is only valid for internal Yang-Mills fields. It is violated for the coframe and the metric. Therefore the Kaniel-Itin model is based on somewhat shaky foundations.

In the light of the results mentioned so far, the following questions come to mind: (i) What is the source on the right hand side of the field equation (1)? (ii) Can the Yilmaz-Rosen metric also be adjusted to the trace part of (1) and, more generally, to a possible source term on the right hand side of (1)? (iii) Is there a consistent variational principle available which would allow to derive (1), including a source term, from a suitable Lagrangian? (iv) What is the (geometrical?) meaning of the constrained variations of Kaniel & Itin?

The purpose of this article is to try to answer these questions. Moreover, along our way, we will discuss some unclear points on the commutativity of variation and Hodge star which led to some (so far uncorrected) mistakes in the literature.—

In Sec. 2 we provide some background material on how to derive wave equations of the general type (1) from Lagrangians in Maxwell's theory and in theories of other internal fields. Here and in the following *internal* fields are those which do not depend on the spacetime geometry (in contrast to ϑ^α and $g_{\alpha\beta}$). In this way we are able to understand how the Lagrangian of Kaniel-Itin comes up in the first place. But since they identify the *gravitational potential* with the *coframe*, we run into trouble from the point of view of finding a suitable Lagrangian.

Any reasonable gauge approach to gravity contains in some way the gauging of the translation group. The simplest gauge theories of gravity are teleparallel theories with only the translation group as gauge group. They already require the knowledge of how to vary the Hodge dual of forms. In a teleparallel theory, spacetime can be described by an *orthonormal* coframe ϑ^α as the only gravitational field variable, which is interpreted as translational gauge potential, see [5]. And this gauge potential was used by Kaniel-Itin in their model. Accordingly, their Lagrangian is a special teleparallelism Lagrangian with the additional postulate of constrained variations. The commutativity of δ and * is, in general, *not* fulfilled for gauge theories of *external* (or spacetime) groups, i.e., for gravitational gauge theories. In this case it is important to know the commutator $\delta^\star - {}^\star\delta$ of the variation δ and the Hodge star *.

Therefore, in Sec. 3 we will derive the master formula (33) for $\delta^\star - {}^\star\delta$. We will include general variations of the components $g_{\alpha\beta}$ of the Riemannian metric g besides those of a (not necessarily orthonormal) coframe ϑ^α. If we *insist*, in accordance with the Kaniel-Itin postulate, on commutativity of δ and *, then the variations $\delta g_{\alpha\beta}$ of the components of the metric are no longer independent and can be expressed in terms of the variation $\delta\vartheta^\alpha$ of the coframe, see (35).

In Sec. 4 we give a short overview of teleparallelism theories and the relevant quadratic Lagrangians. We will discuss the viable set of Lagrangians and display the results in Table 1. We will show that the KI-Lagrangian, for arbitrary variations, is not viable. Some errors in the literature (see Schweitzer et al. [6, 7]) are rectified.

In Sec. 5, we evaluate the model of Kanin & Itin [2]. The field equation of the constrained variational principle is the *antisymmetric part* of a wave equation for ϑ^α, in contrast to the full wave equation as claimed by Kaniel and Itin.

The Yilmaz-Rosen metric, found by Yilmaz [3, Eqs.(18) and (20)] in 1958 as a solution in the context of a scalar field theory of gravitation, also turned out to be a solution of the bi-metric theory of gravitation of Rosen [4];

cf. also [8, 9]. And, in the Kaniel-Itin model, it solves the tracefree wave equation. In Sec. 5.3 we compare the Yilmaz-Rosen with the Schwarzschild metric and give, in Sec. 5.4, a motivation for the emergence of the Yilmaz-Rosen metric. Finally, we investigate the implications that would arise if the Yilmaz-Rosen metric is considered to be a solution of the field equation of Kaniel & Itin including its trace. In Sec. 6 we collect our arguments.

2 Prolegomena to the Kaniel-Itin model

2.1 Maxwell's theory and the wave equation

The kinetic part of the Lagrangian of a Yang-Mills theory is conventionally built from the first derivative of the gauge potential A and the corresponding Hodge dual. For an internal gauge group, such as for the $U(1)$ or the $SU(2)$, the gauge potential A is *independent* of the metric g or the coframe ϑ^α of the underlying spacetime manifold. Then the variation δ of A commutes with the Hodge star operator *. Let us illustrate this for Maxwell's theory, i.e., for $U(1)$-gauge theory in Minkowski spacetime.

The Maxwell Lagrangian is given by[1]

$$L_{\rm Max} = \frac{1}{2}\, dA \wedge {}^\star dA\,. \tag{6}$$

The variation of the 1-form A is independent of the variations $\delta\vartheta^\alpha$ or $\delta g_{\alpha\beta}$; furthermore, it commutes with the exterior derivative, since the variation is defined in this way. Therefore, with the *co*derivative $d^\dagger := -{}^\star d^\star$, we find

$$\delta L_{\rm Max} = d(\delta A \wedge {}^\star dA) - \delta A \wedge {}^\star d^\dagger dA\,. \tag{7}$$

Thus the vacuum field equation reads:

$$-{}^\star d^\dagger dA = 0\,. \tag{8}$$

Additionally, we take the Lorentz condition

$$d^\dagger A = 0 \tag{9}$$

as a gauge condition. Then, introducing the d'Alembertian

$$\Box := d^\dagger d + dd^\dagger = -{}^\star d^\star d - d^\star d^\star\,, \tag{10}$$

[1]We are using the calculus of exterior differential forms, cf. [10, 11]. Our conventions are fixed in [12].

the vacuum field equation can be rewritten as

$$-\Box\,{}^{\star}A = 0\,. \tag{11}$$

One could try to derive (11) directly by supplementing (6) with a suitable Lagrangian. The choice

$$L^{\dagger}_{\rm Max} := \frac{1}{2}\, d^{\dagger}A \wedge {}^{\star}d^{\dagger}A \tag{12}$$

looks suggestive. It leads to the field equation

$$-{}^{\star}d\, d^{\dagger}A = 0\,, \tag{13}$$

which should be compared with (8). Consequently the sum of the Lagrangians (6) and (12), enriched by a matter Lagrangian,

$$L_{\rm Max} + L^{\dagger}_{\rm Max} + L_{\rm mat} = \frac{1}{2}\left(dA \wedge {}^{\star}dA + d^{\dagger}A \wedge {}^{\star}d^{\dagger}A\right) + L_{\rm mat} \tag{14}$$

would yield directly the wave equation:

$$\Box\,{}^{\star}A = \frac{\delta L_{\rm mat}}{\delta A}\,. \tag{15}$$

However, the Lagrangian (12) is not gauge-invariant: For the regauging by means of the arbitrary function f,

$$A \longrightarrow A + df\;, \tag{16}$$

one finds[2]

$$L^{\dagger}_{\rm Max} \longrightarrow L^{\dagger}_{\rm Max} + {}^{\star}d^{\dagger}df \wedge \left(d^{\dagger}A + \frac{1}{2}d^{\dagger}df\right)\;. \tag{17}$$

Accordingly, the Lagrangian (14) has to be rejected. We can obtain the wave equation (11) only in the *special* gauge (9), *after* the derivation of the field equation by means of the variational principle with the Lagrangian (6).

[2]Incidentally, as pointed out by Obukhov, the Lagrangian (12) represents an example for a Lagrangian of a gauge theory of p-forms, see [13, 14]. Then, instead of (16), one has $A \rightarrow A + d^{\dagger}\phi$ as gauge transformation, since $d^{\dagger}d^{\dagger} = 0$. And the new "Lorentz condition" is $dA = 0$.

2.2 A quartet of massive one-form fields

If we used massive fields, then we would have no difficulties with lack of gauge invariance, because the mass term is not gauge invariant anyway. Since we want to study gravity à la Kaniel-Itin, we start with a quartet of 1-form fields k^I, where I is an internal index with $I = \check{0}, \check{1}, \check{2}, \check{3}$. We again derive a wave type equation as in the last subsection, but we now add a massive term for each of the four fields:

$$L_k = \frac{1}{2}\left(dk^I \wedge {}^\star dk^I + \; d^\dagger k^I \wedge {}^\star d^\dagger k^I - m_{(I)}\, k^I \wedge {}^\star k^I\right) + L_{\rm mat}\,. \tag{18}$$

We vary with respect to k^I and find as the Euler-Lagrange equation:

$$\left(\Box + m_{(I)}\right) {}^\star k^I = \frac{\delta L_{\rm mat}}{\delta k^I}\,. \tag{19}$$

One could also think of an additional Higgs-type (or ‘cosmological’) term. Then we would have

$$\begin{aligned} L_{k'} = \frac{1}{2}\left(dk^I \wedge {}^\star dk^I \; + d^\dagger k^I \wedge {}^\star d^\dagger k^I - m_{(I)}\, k^I \wedge {}^\star k^I\right) \\ - \frac{\lambda}{4!}\, \epsilon_{IJKL}\, k^I \wedge k^J \wedge k^K \wedge k^L + L_{\rm mat}\,, \end{aligned} \tag{20}$$

and, as field equation,

$$\left(\Box + m_{(I)}\right) {}^\star k^I + \frac{\lambda}{3!}\, \epsilon_{IJKL}\, k^J \wedge k^K \wedge k^L = \frac{\delta L_{\rm mat}}{\delta k^I}\,. \tag{21}$$

This is as near as we can approach the field equation (1). Since currents are 3-forms, we take the Hodge dual of (1) and remember ${}^\star\,\Box = \Box\,{}^\star$. Furthermore we put a source term on its right hand side. In gravitational theory, this can be only the matter current Σ_α, representing the energy-momentum flux of matter. Then the completed Kaniel-Itin field equation reads:

$$[\Box + \lambda(x)]\, {}^\star\vartheta_\alpha = \frac{\delta L_{\rm mat}}{\delta \vartheta^\alpha} =: \Sigma_\alpha\,. \tag{22}$$

Since $\lambda(x)$ is a *function*, it cannot be identified with some constant mass $m_{(I)}$. Also an interpretation of $\lambda(x)$ as a cosmological constant is obviously meaningless. Therefore the equations (21) and (22) have to be carefully distinguished. In future, we will refer to (22) as the (completed) Kaniel-Itin field equation. Eq.(22) represents the heart of their theory.

2.3 Relation to teleparallel theories

Kaniel-Itin proposed in [2, Eq.(8)] the following Lagrangian for the derivation of the vacuum version of (22):

$$\overset{\text{orig}}{V}_{\text{KI}} = \frac{1}{2}\left[d\vartheta^\alpha \wedge {}^\star d\vartheta_\alpha - d^\dagger\vartheta^\alpha \wedge {}^\star d^\dagger\vartheta_\alpha + \lambda(x)(\vartheta^\alpha \wedge {}^\star\vartheta_\alpha - 4\eta)\right] . \qquad (23)$$

We used our notation here. We want to correct this Lagrangian in two respects: (i) If we compare (20) with (21), it is clear that we should *add* the first two terms, instead of subtracting them. (ii) The Lagrange multiplier term is a hoax since the expression multiplying $\lambda(x)$ *vanishes* identically: We have quite generally $\vartheta^\alpha \wedge \eta_\beta = \delta^\alpha_\beta\eta$, with $\eta_\beta := {}^\star\vartheta_\beta$, and the trace of this equation proves our contention. Taking care of both objections, we will call

$$V_{\text{KI}} \;=\; \frac{1}{2}\left(d\vartheta^\alpha \wedge {}^\star d\vartheta_\alpha + d^\dagger\vartheta^\alpha \wedge {}^\star d^\dagger\vartheta_\alpha\right) \qquad (24)$$

the (corrected) Kaniel-Itin Lagrangian. If we recall that we were able to derive (21) from (20) only because the k^I was an internal field, the variation of which commutes with the star, then it becomes clear that Kaniel-Itin field equation *(22)*, for $\Sigma_\alpha = 0$, *is not the Euler-Lagrange equation of (24)*.

In the Kaniel-Itin model, the Hodge star no longer commutes with the variation $\delta\vartheta^\alpha$ since their gravitational potential ϑ^α is inseparably connected to the spacetime manifold. A postulate of Kaniel-Itin to the opposite, see [2, statement between Eqs.(7) and (8)], is without foundation, at least from a geometrical point of view. Consequently, we will give up this postulate. The Kaniel-Itin Lagrangian belongs to the so-called teleparallelism models of gravity. We will come back to this in Sec. 4.

The addition of the adjoint piece in (24) does not break the translational invariance since $d^\star\vartheta^\alpha$, and thus $d^\dagger\vartheta^\alpha$, can be rewritten in terms of $d\vartheta^\alpha$ (shown here for an orthonormal coframe):

$$d\,{}^\star\vartheta^\alpha = d\eta^\alpha = d\vartheta_\beta \wedge \eta^{\alpha\beta} \, . \qquad (25)$$

An analogous procedure is *not* available in the Maxwellian case for $d^\star A$, that is, $d^\star A$ cannot be expressed in terms of dA. Therefore $L^\dagger_{\text{Max}}$ is not gauge invariant, see (17), and has to be rejected as a decent Lagrangian.

3 Variation of the Hodge dual of a form

3.1 The master formula

In order to install the *variation* δ as a derivation, we demand that it fulfills an even Leibniz rule,

$$\delta(\omega_1 \wedge \omega_2) = \delta\omega_1 \wedge \omega_2 + \omega_1 \wedge \delta\omega_2 \,, \tag{26}$$

where ω_1 and ω_2 are arbitrary exterior differential forms. The Leibniz rule is even, because the variation does not change the degree of the form. In contrast to this, the interior product $v \rfloor$ (here v is a vector) and the exterior derivative d decrease or increase, respectively, the degree of the form by one and fulfill an *odd* Leibniz rule.

Furthermore, we need a relation between the variation and the exterior derivative. According to the definition of the variation, they simply commute:

$$[d, \delta] = 0 \,. \tag{27}$$

Let us now turn to the Hodge star operator, see [12]. It maps a p–form $\psi = \frac{1}{p!}\,\psi_{\alpha_1\cdots\alpha_p}\vartheta^{\alpha_1}\wedge\cdots\wedge\vartheta^{\alpha_p}$ into an $(n-p)$–form ${}^\star\psi$; here n is the dimension of the manifold, i.e., in our case $n = 4$. In terms of components we have

$$\begin{aligned} {}^\star\psi := \frac{1}{(n-p)!\,p!}\sqrt{|\det g_{\mu\nu}|}\, g^{\alpha_1\gamma_1}\cdots g^{\alpha_p\gamma_p} \;\times \\ \epsilon_{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_{n-p}}\,\psi_{\gamma_1\cdots\gamma_p}\,\vartheta^{\beta_1}\wedge\cdots\wedge\vartheta^{\beta_{n-p}} \,, \end{aligned} \tag{28}$$

where ϵ is the Levi-Civita symbol. Besides the ϑ-basis $\Big\{1, \vartheta^{\alpha_1}, \vartheta^{\alpha_1}\wedge\vartheta^{\alpha_2}, \ldots, \vartheta^{\alpha_1}\wedge\vartheta^{\alpha_2}\wedge\cdots\wedge\vartheta^{\alpha_n}\Big\}$, having the Hodge star at our disposal, we may define the so-called η-basis:

$$\begin{aligned} \Big\{\eta, \eta^{\alpha_1}, \eta^{\alpha_1\alpha_2}, \ldots, \eta^{\alpha_1\alpha_2\cdots\alpha_n}\Big\} := \\ \Big\{{}^\star 1, {}^\star\vartheta^{\alpha_1}, {}^\star(\vartheta^{\alpha_1}\wedge\vartheta^{\alpha_2}), \ldots, {}^\star(\vartheta^{\alpha_1}\wedge\vartheta^{\alpha_2}\wedge\cdots\wedge\vartheta^{\alpha_n})\Big\} \,. \end{aligned} \tag{29}$$

Now we can derive the desired expression for $\delta^\star\phi$, for an arbitrary p-form ϕ. We can saturate the $(n-p)$-form ${}^\star\phi$ with coframes ϑ^β such as to arrive at the n-form

$$\vartheta^{\beta_1}\wedge\cdots\wedge\vartheta^{\beta_p}\wedge{}^\star\phi \overset{(97)}{=} \phi\wedge{}^\star\left(\vartheta^{\beta_1}\wedge\cdots\wedge\vartheta^{\beta_p}\right) \overset{(100\text{b})}{=} \phi\wedge\eta^{\beta_1\cdots\beta_p} \,. \tag{30}$$

We vary (30). Then the even Leibniz rule (26) for the variation leads to

$$\delta\left(\vartheta^{\beta_1}\wedge\cdots\wedge\vartheta^{\beta_p}\right)\wedge{}^\star\phi+\vartheta^{\beta_1}\wedge\cdots\wedge\vartheta^{\beta_p}\wedge\delta^\star\phi = \delta\phi\wedge\eta^{\beta_1\cdots\beta_p}+\phi\wedge\delta\eta^{\beta_1\cdots\beta_p}\,. \tag{31}$$

Thus, apparently, we know how to vary the Hodge star, provided we know how to vary the η-basis. The variation of the $(n-p)$-form $\eta^{\alpha_1\cdots\alpha_p}$ is computed in the appendix in Sec. A.2. It turns out to be

$$\delta\eta^{\beta_1\dots\beta_p}=\delta\vartheta^\mu\wedge\left(e_\mu\rfloor\eta^{\beta_1\dots\beta_p}\right)+\delta g_{\kappa\lambda}\left(\vartheta^{(\kappa|}\wedge\eta^{\beta_1\dots\beta_p|\lambda)}-\frac{1}{2}\,g^{\kappa\lambda}\eta^{\beta_1\dots\beta_p}\right). \tag{32}$$

Incidentally, for the special choice of an orthonormal tetrad, we have $\delta g_{\alpha\beta}=0$, and the last two terms vanish; in particular, we then have $\delta\eta^{\beta_1\cdots\beta_n}=0$. But we will not introduce this specialization at the present stage.

If we resolve (31) with respect to $\delta^\star\phi$, then, after some intermediary algebra, see Sec. A.3 of the appendix, we find for arbitrary p-forms ϕ the master formula:

$$\boxed{\begin{aligned}(\delta^\star-{}^\star\delta)\,\phi = {}&\delta\vartheta^\alpha\wedge(e_\alpha\rfloor{}^\star\phi)-{}^\star\left[\delta\vartheta^\alpha\wedge(e_\alpha\rfloor\phi)\right]\\ &+\delta g_{\alpha\beta}\left[\vartheta^{(\alpha}\wedge(e^{\beta)}\rfloor{}^\star\phi)-\frac{1}{2}\,g^{\alpha\beta}\,{}^\star\phi\right].\end{aligned}} \tag{33}$$

Again, for orthonormal (co)frames, the terms in the second line vanish since $\delta g_{\alpha\beta}=0$.

3.2 Constrained variations à la Yang-Mills

Proposition: *The condition*

$$\delta^\star\phi={}^\star\delta\phi \tag{34}$$

for an arbitrary p-form ϕ is equivalent to the following relation between the variation of the metric and the coframe:

$$\delta g_{\alpha\beta}=-2g_{\gamma(\alpha}\,e_{\beta)}\rfloor\delta\vartheta^\gamma=-2\omega_{(\alpha\beta)}\,,\qquad \textit{with}\qquad \delta\vartheta^\gamma=\omega_\delta{}^\gamma\vartheta^\delta\,. \tag{35}$$

Therefore, for an orthonormal coframe, the allowed variations are of the Lorentz type, i.e., $\omega_{(\alpha\beta)}\equiv 0$.

To prove[3] this equivalence we first assume that (34) is valid. We apply this constrained variation to the volume n-form $\eta:={}^\star 1$, defined in (29),

$$\delta\eta=\delta({}^\star 1)={}^\star(\delta 1)\equiv 0\;, \tag{36}$$

[3]It was Yuri Obukhov who suggested essential parts of this proof to us.

since the constant 1 is not varied. In turn, for the identity

$$\vartheta^\alpha \wedge \eta_\beta = \delta^\alpha_\beta \, \eta \tag{37}$$

we find:

$$\delta\vartheta^\alpha \wedge \eta_\beta + \vartheta^\alpha \wedge \delta\eta_\beta = 0 \, . \tag{38}$$

The commutation rule (34) applied to ϑ_β yields

$$\delta\eta_\beta = \delta({}^\star\vartheta_\beta) = {}^\star(\delta\vartheta_\beta) \, . \tag{39}$$

Thus, by exterior multiplication with ϑ^α we arrive at

$$\vartheta^\alpha \wedge \delta\eta_\beta = \vartheta^\alpha \wedge {}^\star(\delta\vartheta_\beta) = \delta\vartheta_\beta \wedge {}^\star\vartheta^\alpha = \delta(g_{\beta\gamma}\vartheta^\gamma) \wedge \eta^\alpha \, . \tag{40}$$

On substitution into (38), we find

$$\delta\vartheta^\alpha \wedge \eta_\beta + g_{\beta\gamma}\, \delta\vartheta^\gamma \wedge \eta^\alpha + \delta g_{\beta\gamma}\, \vartheta^\gamma \wedge \eta^\alpha = 0 \tag{41}$$

or, since $\eta_\beta = e_\beta \rfloor \eta$,

$$\delta g_{\alpha\beta} = -2 g_{\gamma(\alpha}\, e_{\beta)} \rfloor \delta\vartheta^\gamma \, . \tag{42}$$

The 1-form $\delta\vartheta^\alpha$ can be expanded with respect to the coframe:

$$\delta\vartheta^\gamma = \omega_\delta{}^\gamma \vartheta^\delta \, . \tag{43}$$

We insert (43) into (42). Then we find

$$\delta g_{\alpha\beta} = -2\omega_{(\alpha\beta)} \, . \tag{44}$$

To investigate the reverse part of the proposition, we apply the general rule (33) for the variations of Hodge dual forms and use (43) and (44):

$$\begin{aligned}
(\delta^\star - {}^\star\delta)\, \phi = {} & \omega^{\beta\alpha}\Big[\vartheta_\beta \wedge (e_\alpha \rfloor {}^\star\phi) - {}^\star\big[\vartheta_\beta \wedge (e_\alpha \rfloor \phi)\big]\Big] \\
& - 2\omega^{(\beta\alpha)}\, \vartheta_\alpha \wedge (e_\beta \rfloor {}^\star\phi) + \omega_\gamma{}^\gamma\, {}^\star\phi \\
= {} & \omega^{\beta\alpha}\Big[- \vartheta_\alpha \wedge (e_\beta \rfloor {}^\star\phi) - {}^\star\big[\vartheta_\beta \wedge (e_\alpha \rfloor \phi)\big] + g_{\alpha\beta}\, {}^\star\phi\Big] \\
= {} & \omega^{\beta\alpha}\Big[e_\beta \rfloor (\vartheta_\alpha \wedge {}^\star\phi) - {}^\star\big[\vartheta_\beta \wedge (e_\alpha \rfloor \phi)\big]\Big] \\
\overset{(98a),\,(98c)}{=} {} & \omega^{\beta\alpha}\, e_\beta \rfloor {}^\star(e_\alpha \rfloor \phi)\Big[\frac{1}{(-1)^{p-1}} - (-1)^{p-1}\Big] = 0 \, .
\end{aligned} \tag{45}$$

Thus the proposition is proved.

4 Teleparallelism theories of gravity

A Minkowski space is invariant under rigid translations. In order to make a manifold locally translation invariant, one can introduce a gauge potential by means of a dynamical coframe ϑ^α, see [15, 5, 16, 17, 18, 11, 19]. Such a spacetime carries a torsion, but no curvature: It is a so-called *Weitzenböck spacetime*. Then, by picking a suitable frame, the connection $\Gamma_\alpha{}^\beta$ can always globally be transformed to zero.

Therefore, in a teleparallel theory, the coframe ϑ^α is the basic gravitational field variable. Furthermore, let a metric g be given of Minkowskian signature, i. e. of index 3 (number of negative eigenvalues of the metric). For the rest of this paper we choose the coframe to be *orthonormal*, $g := o_{\alpha\beta}\, \vartheta^\alpha \otimes \vartheta^\beta$, with $o_{\alpha\beta} = \mathrm{diag}(+1, -1, -1, -1)$, and raise and lower the frame indices by means of $o_{\alpha\beta}$.

4.1 The Rumpf Lagrangians

According to Rumpf [20], a general quadratic Lagrangian for the coframe ϑ^α can be expanded in terms of the *gauge-invariant* translational Lagrangians (ℓ = Planck length, Λ = cosmological constant $= \rho_0/2$):

$$V = \frac{1}{2\ell^2} \sum_{K=0}^{4} \rho_K \; {}^{[K]}V \;, \tag{46}$$

with

$${}^{[0]}V = \frac{1}{4}\, \vartheta^\alpha \wedge {}^\star\vartheta_\alpha = \eta \;, \tag{47a}$$

$${}^{[1]}V = d\vartheta^\alpha \wedge {}^\star d\vartheta_\alpha \qquad \text{(pure Yang-Mills type)} \;, \tag{47b}$$

$${}^{[2]}V = \Big(d\vartheta_\alpha \wedge \vartheta^\alpha\Big) \wedge {}^\star \left(d\vartheta_\beta \wedge \vartheta^\beta\right) \;, \tag{47c}$$

$${}^{[3]}V = \left(d\vartheta^\alpha \wedge \vartheta^\beta\right) \wedge {}^\star \left(d\vartheta_\alpha \wedge \vartheta_\beta\right) = d\vartheta^\alpha \wedge \vartheta^\beta \wedge \left(e_\beta \rfloor {}^\star d\vartheta_\alpha\right) = 2\, {}^{[1]}V \;, \tag{47d}$$

$${}^{[4]}V = \left(d\vartheta_\alpha \wedge \vartheta^\beta\right) \wedge {}^\star \Big(d\vartheta_\beta \wedge \vartheta^\alpha\Big) \;. \tag{47e}$$

Since ${}^{[3]}V = 2\,{}^{[1]}V$, we can always put $\rho_3 = 0$.

4.2 The irreducible Lagrangians

Alternatively, the field strength $d\vartheta^\alpha$ can be decomposed into three pieces which transform irreducibly under the Lorentz group:

$$d\vartheta^\alpha \;=\; {}^{(1)}d\vartheta^\alpha + {}^{(2)}d\vartheta^\alpha + {}^{(3)}d\vartheta^\alpha \,. \tag{48}$$

Here we defined (in parentheses we are mentioning the corresponding names of our computer algebra programs):

$$ {}^{(1)}d\vartheta^\alpha := d\vartheta^\alpha - {}^{(2)}d\vartheta^\alpha - {}^{(3)}d\vartheta^\alpha \qquad \texttt{(tentor),} \tag{49a}$$

$$ {}^{(2)}d\vartheta^\alpha := \frac{1}{3}\vartheta^\alpha \wedge \left(e_\beta \rfloor d\vartheta^\beta\right) \qquad \texttt{(trator),} \tag{49b}$$

$$ {}^{(3)}d\vartheta^\alpha := \frac{1}{3} e_\alpha \rfloor \left(\vartheta^\beta \wedge d\vartheta_\beta\right) \qquad \texttt{(axitor).} \tag{49c}$$

In terms of the numbers of components involved, we have the decomposition $24 = 16 \oplus 4 \oplus 4$.

Then we can write

$$V = \frac{1}{2\ell^2}\left[a_0\,\eta + \sum_{I=1}^{3} a_I \left(d\vartheta^\alpha \wedge {}^\star\,{}^{(I)}d\vartheta_\alpha\right)\right] \,. \tag{50}$$

We substitute (48) and (49) into (47). Then a comparison between (46) and (50) yields

$$\rho_1 = \frac{1}{3}\,(a_2 + 2a_1)\ , \qquad \rho_2 = \frac{1}{3}\,(a_3 - a_1)\ , \qquad \rho_4 = \frac{1}{3}\,(a_1 - a_2)\ , \tag{51}$$

$$a_1 = \rho_1 + \rho_4\ , \qquad a_2 = \rho_1 - 2\rho_4\ , \qquad a_3 = \rho_1 + 3\rho_2 + \rho_4\ , \tag{52}$$

and, additionally $a_0 = \rho_0 = 2\Lambda$. These relations were checked by means of a computer algebra program, see Sec. B of the appendix.

4.3 Field equation

The field equation of a general translation invariant Lagrangian reads [15]

$$dH_\alpha - E_\alpha = \Sigma_\alpha\ , \tag{53}$$

with

$$H_\alpha := -\frac{\partial V}{\partial d\vartheta^\alpha}\ , \quad E_\alpha := \frac{\partial V}{\partial \vartheta^\alpha}\ , \quad \text{and} \qquad \Sigma_\alpha := \frac{\delta L_{\rm mat}}{\delta\vartheta^\alpha}\,. \tag{54}$$

In (54), the partial derivatives are implicitly defined by means of the variation of the Lagrangian:

$$\delta V = \delta\vartheta^\alpha \wedge \frac{\partial V}{\partial \vartheta^\alpha} + \delta\, d\vartheta^\alpha \wedge \frac{\partial V}{\partial d\vartheta^\alpha}\,. \tag{55}$$

If we use the abbreviations (54), we don't need our master formula for the computation of the field equation. Alternatively, one can take the Lagrangian (46) together with (47) and vary the resulting expression by using (33) with $\delta g_{\alpha\beta} = 0$. This yields the explicit form of the field equation (53), cf. Kopczyński [21][4]:

$$\begin{aligned}
-2\,\ell^2\Sigma_\alpha = {} & 2\rho_1\, d\,{}^\star d\vartheta_\alpha - 2\rho_2\, \vartheta_\alpha \wedge d\,{}^\star \left(d\vartheta^\beta \wedge \vartheta_\beta\right) - 2\rho_4\, \vartheta_\beta \wedge d\,{}^\star \left(\vartheta_\alpha \wedge d\vartheta^\beta\right) \\
& + \rho_1 \Big[e_\alpha \rfloor \left(d\vartheta^\beta \wedge {}^\star d\vartheta_\beta\right) - 2\left(e_\alpha \rfloor d\vartheta^\beta\right) \wedge {}^\star d\vartheta_\beta \Big] \\
& + \rho_2\, \Big[2 d\vartheta_\alpha \wedge {}^\star \left(d\vartheta^\beta \wedge \vartheta_\beta\right) \\
& \quad + e_\alpha \rfloor \left(d\vartheta^\gamma \wedge \vartheta_\gamma \wedge {}^\star \left(d\vartheta^\beta \wedge \vartheta_\beta\right)\right) - 2\left(e_\alpha \rfloor d\vartheta^\beta\right) \wedge \vartheta_\beta \wedge {}^\star \left(d\vartheta^\gamma \wedge \vartheta_\gamma\right) \Big] \\
& + \rho_4 \Big[2 d\vartheta_\beta \wedge {}^\star \left(\vartheta_\alpha \wedge d\vartheta^\beta\right) \\
& \quad + e_\alpha \rfloor \left(\vartheta_\gamma \wedge d\vartheta^\beta \wedge {}^\star \left(d\vartheta^\gamma \wedge \vartheta_\beta\right)\right) - 2\left(e_\alpha \rfloor d\vartheta^\beta\right) \wedge \vartheta_\gamma \wedge {}^\star \left(d\vartheta^\gamma \wedge \vartheta_\beta\right) \Big]\,.
\end{aligned} \tag{56}$$

In the first line of this equation we displayed the leading terms containing second derivatives of the coframe. In the remaining terms there enter only first derivatives.

4.4 Decomposition of the Kaniel-Itin Lagrangian

In order to better recognize the structure of the KI-Lagrangian, one can decompose it in its irreducible pieces as well as into the Rumpf Lagrangians.

The first term of the KI-Lagrangian (24) is exactly the Yang-Mills type Lagrangian ${}^{[1]}V$, cf. with (47). The term $d^\dagger\vartheta^\alpha \wedge {}^\star d^\dagger \vartheta_\alpha = -d^\star\vartheta^\alpha \wedge {}^\star d^\star \vartheta_\alpha$ is left for scrutiny. Using formula (25) and other rules, see, e.g., [12] and Sec.

[4]Kopczyński denoted the Rumpf Lagrangians by K. We have the following translation rules: $K^1 = {}^{[4]}V$, $K^2 = {}^{[2]}V$, $K^3 = {}^{[1]}V$. In (56), the second derivatives of the coframe are exactly the same (for $\ell^2 = 1$) as those in the corresponding three equations of Kopczyński [21, top of p. 503].

A.1 for details, we find

$$
\begin{aligned}
-d\,{}^\star\vartheta^\alpha\wedge{}^\star d\,{}^\star\vartheta_\alpha &= -d\eta^\alpha\wedge{}^\star d\eta_\alpha = -d\vartheta_\beta\wedge\eta^{\alpha\beta}\wedge{}^\star\left(d\vartheta^\gamma\wedge\eta_{\alpha\gamma}\right)\\
&= -\vartheta^\alpha\wedge\vartheta^\beta\wedge{}^\star d\vartheta_\beta\wedge{}^\star\left(\vartheta_\alpha\wedge\vartheta_\gamma\wedge{}^\star d\vartheta^\gamma\right)\\
&= \vartheta^\alpha\wedge\vartheta^\beta\wedge{}^\star d\vartheta_\beta\wedge{}^\star\left[\vartheta_\alpha\wedge{}^\star\left(e_\gamma\rfloor d\vartheta^\gamma\right)\right]\\
&= \vartheta^\beta\wedge{}^\star d\vartheta_\beta\wedge\vartheta^\alpha\wedge\left[e_\alpha\rfloor\left(e_\gamma\rfloor d\vartheta^\gamma\right)\right]\\
&= {}^\star d\vartheta_\beta\wedge\vartheta^\beta\wedge\left(e_\gamma\rfloor d\vartheta^\gamma\right) = {}^\star\left[\vartheta^\beta\wedge\left(e_\gamma\rfloor d\vartheta^\gamma\right)\right]\wedge d\vartheta_\beta\\
&= -d\vartheta_\beta\wedge{}^\star\left[e_\gamma\rfloor\left(\vartheta^\beta\wedge d\vartheta^\gamma\right)\right]+d\vartheta_\gamma\wedge{}^\star d\vartheta^\gamma\\
&= -d\vartheta_\alpha\wedge\vartheta_\beta\wedge{}^\star\left(\vartheta^\alpha\wedge d\vartheta^\beta\right)+d\vartheta^\alpha\wedge{}^\star d\vartheta_\alpha = {}^{[1]}V-{}^{[4]}V\,.
\end{aligned}
\tag{57}
$$

Accordingly, the KI-Lagrangian (24) can be rewritten as

$$
V_{\rm KI}=\frac{1}{2\ell^2}\Big[2\,d\vartheta^\alpha\wedge{}^\star d\vartheta_\alpha-\left(d\vartheta_\alpha\wedge\vartheta^\beta\right)\wedge{}^\star\Big(d\vartheta_\beta\wedge\vartheta^\alpha\Big)\Big]\,, \tag{58}
$$

and we can read off the ρ_K coefficients as follows:

$$
\rho_1=1+1=2\,,\quad \rho_2=0\,,\quad \rho_4=-1\,. \tag{59}
$$

By *subtracting* the adjoint term we would have $\rho_1=\rho_2=0$ and $\rho_4=1$, i.e., we would get the von der Heyde Lagrangian [22].

The coefficients of the decomposition into irreducible pieces, by using (52), turn out to be

$$
a_1=1\,,\quad a_2=4\,,\quad a_3=1 \tag{60}
$$

(in the von der Heyde case, we have $a_1=1$, $a_2=-2$, $a_3=1$). Accordingly, the KI-Lagrangian can be rewritten in the form

$$
V_{\rm KI}=-\frac{1}{2}\,d\vartheta^\alpha\wedge H_\alpha\,, \tag{61}
$$

with the translational “excitation”

$$
H_\alpha=-\frac{1}{\ell^2}\,{}^\star\Big(a_1\,{}^{(1)}d\vartheta_\alpha+a_2\,{}^{(2)}d\vartheta_\alpha+a_3\,{}^{(3)}d\vartheta_\alpha\Big) \tag{62}
$$

and the coefficients (60).

The Lagrangian $V_{\rm KI}$ is *not* locally Lorentz invariant. Rather, a locally Lorentz invariant theory results from the following choice of the parameters:

$$
a_1\;=\;1\,,\quad a_2\;=\;-2\,,\quad a_3=-\frac{1}{2}\,. \tag{63}
$$

This represents the teleparallel *equivalent* of Einstein’s general relativity.

4.5 Viable Lagrangians

The form of a general quadratic Lagrangian was displayed in Eqs. (50) and (46). Various choices of parameters a_I or ρ_K correspond to various teleparallel theories of gravity. We call a specific Lagrangian *viable* if it leads to a theory which fulfills the following conditions: (i) It has the correct Newtonian approximation. (ii) It agrees with the first post-Newtonian approximation of general relativity. (iii) It has the Schwarzschild metric as exact solution in the case of spherical symmetry.

The question which parameters yield a viable Lagrangian has already been discussed in the literature, see for example [23, 7, 21, 11]. The result is that we have viable Lagrangians for $a_1 = 1\,,\ a_2 = -2\,,\ a_3 =$ arbitrary or $\rho_4 = 1\,,\ \rho_1 = 0\,,\ \rho_2 =$ arbitrary. The arbitrary a_3 or ρ_2 pieces, respectively, represents the axial square contribution $A \wedge {}^\star A$ of the torsion, with $A := \frac{1}{3}\vartheta^\beta \wedge d\vartheta_\beta$ and ${}^{(3)}d\vartheta^\alpha = e_\alpha \rfloor A$. Deviations between viable theories due to different axial pieces only show up in fifth order of the post-Newtonian approximation [23]. Therefore, on a phenomenological level, all viable teleparallel theories are indistinguishable. In Table 1 we have listed some quadratic torsion Lagrangians.

It follows from the end of the last subsection, see also (63), that only the teleparallel equivalent of Einstein's general relativity is both viable and locally Lorentz invariant. It has not yet been clearly answered if in this context local Lorentz invariance is obligatory or merely an aesthetic feature.

	$\mathrm{GR}_{\parallel}$ [5]	vdH [22]	viable	YM	YM†	KI [2]
a_1	1	1	1	1	0	1
a_2	-2	-2	-2	1	3	4
a_3	$-\frac{1}{2}$	1	arb.	1	0	1
ρ_1	0	0	0	1	1	2
ρ_2	$-\frac{1}{2}$	0	arb.	0	0	0
ρ_4	1	1	1	0	-1	-1

Table 1: *This table lists the a_I and the ρ_K coefficients for different teleparallel Lagrangians. $GR_{\parallel}$, spelled out in the first column, represents a viable gravitational model, the same is true for the von der Heyde case. Obviously, the Kaniel-Itin Lagrangian, in the framework of the conventional variational procedure, is not viable. We have $KI = YM + YM^\dagger$ and $vdH = YM - YM^\dagger$.*

4.6 Schweizer-Straumann-Wipf amended

Schweizer and Straumann [6] and Schweizer, Straumann, and Wipf [7] investigated the von der Heyde teleparallelism Lagrangian [22], see also [23]. In particular they showed that, to first post-Newtonian order, the von der Heyde theory predicts the same gravitational radiation loss as general relativity. However, they assumed in some places the incorrect commutation rule $\delta\,{}^\star = {}^\star\delta$. Therefore some equations in these articles must be corrected. We stress that the corrections do not influence their overall results, though.

We present the corrected formulas of the article by Schweizer, Straumann, and Wipf [7] in the numbering used there, but in our notation. The correcting terms are printed in bold.

The wrong variation first shows up in the explicit expressions of the canonical energy-momentum tensors derived from the teleparallel version of the Hilbert-Einstein Lagrangian, ϵ^E_α, and the difference between the von der Heyde Lagrangian and the teleparallel version of the Hilbert-Einstein Lagrangian, $\Delta\epsilon_\alpha$:

$$
\begin{aligned}
\epsilon^E_\alpha = & -d\left[\vartheta^\beta \wedge {}^\star(d\vartheta_\beta \wedge \vartheta_\alpha)\right] - d\vartheta^\beta \wedge {}^\star(d\vartheta_\alpha \wedge \vartheta_\beta) \qquad (2.12),[7] \\
& + \frac{1}{2} d\left\{\vartheta_\alpha \wedge {}^\star\left(d\vartheta^\beta \wedge \vartheta_\beta\right)\right\} + \frac{1}{2} d\vartheta_\alpha \wedge {}^\star\left(d\vartheta^\beta \wedge \vartheta_\beta\right) \\
& \boldsymbol{+\frac{1}{2} e_\alpha \rfloor \left(\vartheta_\gamma \wedge d\vartheta^\beta\right) \wedge {}^\star\left(d\vartheta^\gamma \wedge \vartheta_\beta\right)} \\
& \boldsymbol{+\frac{1}{2} d\vartheta^\gamma \wedge \vartheta_\beta \wedge e_\alpha \rfloor {}^\star\left(\vartheta_\gamma \wedge d\vartheta^\beta\right)} \\
& \boldsymbol{-\frac{1}{4} e_\alpha \rfloor \left(d\vartheta^\beta \wedge \vartheta_\beta\right) \wedge {}^\star\left(d\vartheta^\gamma \wedge \vartheta_\gamma\right)} \\
& \boldsymbol{-\frac{1}{4} d\vartheta^\beta \wedge \vartheta_\beta \wedge e_\alpha \rfloor {}^\star\left(d\vartheta^\gamma \wedge \vartheta_\gamma\right)}\,, \\
\Delta\epsilon_\alpha = & \, d\vartheta_\alpha \wedge {}^\star\left(d\vartheta^\beta \wedge \vartheta_\beta\right) - \frac{1}{2}\vartheta_\alpha \wedge d\,{}^\star\left(d\vartheta^\beta \wedge \vartheta_\beta\right) \qquad (2.13),[7] \\
& \boldsymbol{-\frac{1}{4} e_\alpha \rfloor \left(d\vartheta^\beta \wedge \vartheta_\beta\right) \wedge {}^\star\left(d\vartheta^\gamma \wedge \vartheta_\gamma\right)} \\
& \boldsymbol{-\frac{1}{4} d\vartheta^\beta \wedge \vartheta_\beta \wedge e_\alpha \rfloor {}^\star\left(d\vartheta^\gamma \wedge \vartheta_\gamma\right)}\,.
\end{aligned}
$$

Since the additional terms do not influence the antisymmetric part of $\Delta\epsilon_\alpha$, we only need to correct the symmetric part:

$$
\Delta\epsilon^s_\alpha = -\frac{1}{2}\,{}^\star\Big[\left(d\vartheta_\alpha \wedge \vartheta^\beta + d\vartheta^\beta \wedge \vartheta_\alpha\right) \wedge {}^\star\left(d\vartheta^\gamma \wedge \vartheta_\gamma\right) \qquad (3.1),[7]
$$

$$-\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\vartheta^{\beta}\wedge d\vartheta^{\gamma}\wedge\vartheta_{\gamma}\wedge e_{\alpha}}\rfloor^{\star}\left(\boldsymbol{d\vartheta^{\delta}\wedge\vartheta_{\delta}}\right)$$
$$\left.+\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{d\vartheta^{\gamma}\wedge\vartheta_{\gamma}}\wedge^{\star}\left(\boldsymbol{d\vartheta^{\delta}\wedge\vartheta_{\delta}}\right)\boldsymbol{\delta^{\beta}_{\alpha}}\right]\eta_{\beta}\,.$$

Hence we find the corrected field equation as

$$\epsilon^{E}_{\alpha}-\frac{(\lambda-1)}{2}\,{}^{\star}\left[\left(d\vartheta_{\alpha}\wedge\vartheta^{\beta}+d\vartheta^{\beta}\wedge\vartheta_{\alpha}\right)\wedge^{\star}\left(d\vartheta^{\gamma}\wedge\vartheta_{\gamma}\right)\right. \tag{3.3),[7}$$
$$-\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\vartheta^{\beta}\wedge d\vartheta^{\gamma}\wedge\vartheta_{\gamma}\wedge e_{\alpha}}\rfloor^{\star}\left(\boldsymbol{d\vartheta^{\delta}\wedge\vartheta_{\delta}}\right)$$
$$\left.+\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{d\vartheta^{\gamma}\wedge\vartheta_{\gamma}}\wedge^{\star}\left(\boldsymbol{d\vartheta^{\delta}\wedge\vartheta_{\delta}}\right)\boldsymbol{\delta^{\beta}_{\alpha}}\right]\eta_{\beta}=-t^{\alpha}\,.$$

In our units, we have for the λ-parameter of Schweitzer et al. $\lambda=-2\rho_2=\frac{2}{3}\left(1-a_3\right)$. Like the terms discussed in [7], the additional terms are at least quadratic in $\phi_{\alpha\beta}$ (which is the symmetric part of $\Phi_{\alpha\beta}$ in the expansion $\vartheta^{\alpha}=dx^{\alpha}+\Phi^{\alpha}{}_{\beta}\,dx^{\beta}$), such that the arguments of [7, §3] remain unchanged.

The formulas [7, (4.1)] and [7, (4.6)] have to be corrected similarly as the last two equations, replacing $d\vartheta^{\gamma}$ by T^{γ}. Since these explicit expressions are not used in the remainder of [7, §4], the conclusions remain valid therein.

Finally, the quantities $A^{\mu\nu}$, $B^{\mu\nu}$, and $C^{\mu\nu}$ (formulas [7, (5.4b)–(5.4d)]) in the expansion of the quadratic and higher-order terms

$$\epsilon^{\alpha}_{Q}=\Delta\epsilon^{\alpha}_{s}+\left\{A^{\beta\alpha}+B^{\beta\alpha}+C^{\beta\alpha}\right\} \tag{5.4a),[7}$$

would need corrections. Since these quantities are not used explicitly in [7], we did not display the exact expressions here.

5 Kaniel-Itin examined

5.1 Lagrangian

We come back to the Kaniel-Itin Lagrangian (24):

$$V_{\mathrm{KI}}\;=\;\frac{1}{2\ell^{2}}\left(d\vartheta^{\alpha}\wedge{}^{\star}d\vartheta_{\alpha}+d^{\dagger}\vartheta^{\alpha}\wedge{}^{\star}d^{\dagger}\vartheta_{\alpha}\right)\,. \tag{64}$$

Its Euler-Lagrange equation can be read off from (56) by substituting the coefficients (59):

$$\begin{aligned} -2\ell^2\,\Sigma_\alpha &= 2\,\vartheta_\beta\wedge d\,{}^\star\left(\vartheta_\alpha\wedge d\vartheta^\beta\right)+4\,d\,{}^\star d\vartheta_\alpha \\ &\quad + 2e_\alpha\rfloor\left(d\vartheta^\beta\wedge{}^\star d\vartheta_\beta\right)-4\left(e_\alpha\rfloor d\vartheta^\beta\right)\wedge{}^\star d\vartheta_\beta-2d\vartheta_\beta\wedge{}^\star\left(\vartheta_\alpha\wedge d\vartheta^\beta\right) \\ &\quad - e_\alpha\rfloor\left[\vartheta_\gamma\wedge d\vartheta^\beta\wedge{}^\star\left(d\vartheta^\gamma\wedge\vartheta_\beta\right)\right]+2\left(e_\alpha\rfloor d\vartheta^\beta\right)\wedge\vartheta_\gamma\wedge{}^\star\left(d\vartheta^\gamma\wedge\vartheta_\beta\right)\ . \end{aligned} \tag{65}$$

In the first line, we displayed the second derivatives of the gravitational potential. As we already saw in Table 1, this field equation is not viable.

If we use the *the constrained variations* in (64), then we commute δ and the star * and find by simple algebra:

$$\ell^2\,\delta\left(V_{\rm KI}+L_{\rm mat}\right)=\delta\vartheta^\alpha\wedge\left(-\Box\eta_\alpha+\ell^2\,\Sigma_\alpha\right)=0\,. \tag{66}$$

Since the variations are constrained, we have to turn to our proposition and to use (35): $\delta\vartheta^\alpha=\omega_\beta{}^\alpha\,\vartheta^\beta$, with $\omega_{(\alpha\beta)}=0$. On substitution in (66), the field equation turns out to be proportional to the *antisymmetric* part of the wave equation,

$$\vartheta_{[\alpha}\wedge\Box\,\eta_{\beta]}=\ell^2\,\vartheta_{[\alpha}\wedge\Sigma_{\beta]}\ , \tag{67}$$

rather than to the wave equation itself. Accordingly, the constrained variations also lead to a dead end and we have to turn back our attention again to the KI-field equation (22).

5.2 Decomposition of the field equation

Let us split the full wave equation into its different pieces. For that purpose, we most conveniently start from the decomposition of the energy-momentum current as a covector-valued 3-form with 16 independent components:

$$\Sigma_\alpha=\widehat{\not\Sigma}_\alpha+\frac{1}{2}\vartheta_\alpha\wedge\left(e_\gamma\rfloor\Sigma^\gamma\right)+\frac{1}{4}e_\alpha\rfloor\left(\vartheta^\gamma\wedge\Sigma_\gamma\right)\ . \tag{68}$$

Here $\widehat{\not\Sigma}_\alpha$ is its *symmetric traceless* part, the second term on the right hand side its *antisymmetric* and the last term its *trace part*, see [12, eq.(5.1.15)].

We come into better known territory if we decompose Σ_α with respect to the η-basis:

$$\Sigma_\alpha=T^\beta{}_\alpha\,\eta_\beta\ . \tag{69}$$

The $T^{\beta}{}_{\alpha}$'s are the components of the energy-momentum tensor. We can 'saturate' the 3-form Σ_β by means of the 1-form ϑ^α:

$$\begin{aligned}{}^\star\left(\vartheta^\alpha \wedge \Sigma_\beta\right) = {}^\star\left(\vartheta^\alpha \wedge T^\gamma{}_\beta\, \eta_\gamma\right) = T^\gamma{}_\beta\, {}^\star\left(\vartheta^\alpha \wedge \eta_\gamma\right) \\ = T^\alpha{}_\beta\, {}^\star\eta = T^\alpha{}_\beta\, {}^{\star\star}1 = -T^\alpha{}_\beta \end{aligned} \tag{70}$$

or

$$T_{\alpha\beta} = {}^\star\left(\Sigma_\beta \wedge \vartheta_\alpha\right) = e_\alpha \rfloor {}^\star\Sigma_\beta \,. \tag{71}$$

The analog of (68) is, of course, the following splitting of the energy-momentum tensor:

$$T_{\alpha\beta} = \{T_{(\alpha\beta)} - \frac{1}{4}\, T^\gamma{}_\gamma\, g_{\alpha\beta}\} + T_{[\alpha\beta]} + \frac{1}{4}\, T^\gamma{}_\gamma\, g_{\alpha\beta} \,. \tag{72}$$

Coming back to (22), after some algebra, we find the following decomposition:

$$\Box\eta_\alpha - \frac{1}{2}\vartheta_\alpha \wedge (e_\gamma \rfloor \Box\eta^\gamma) - \frac{1}{4} e_\alpha \rfloor\, (\vartheta^\gamma \wedge \Box\eta_\gamma) = \ell^2\, \widehat{\not\Sigma}_\alpha \,, \tag{73a}$$

$$\frac{1}{2}\vartheta_\alpha \wedge \vartheta_\beta \wedge (e_\gamma \rfloor \Box\eta^\gamma) = \ell^2\, \vartheta_{[\alpha} \wedge \Sigma_{\beta]} \,, \tag{73b}$$

$$\vartheta^\gamma \wedge \Box\eta_\gamma + 4\lambda(x)\,\eta = \ell^2\, \vartheta^\gamma \wedge \Sigma_\gamma \,. \tag{73c}$$

We can combine the antisymmetric and the symmetric-tracefree part of the wave equation in order to get its tracefree part. The fine splitting (73) simplifies to the tracefree and the trace part of the field equation (67):

$$\Box\eta_\alpha - \frac{1}{4} e_\alpha \rfloor\, (\vartheta^\gamma \wedge \Box\eta_\gamma) = \ell^2\, \not\Sigma_\alpha \,, \tag{74a}$$

$$\vartheta^\gamma \wedge \Box\eta_\gamma + 4\lambda(x)\,\eta = \ell^2\, \vartheta^\gamma \wedge \Sigma_\gamma \,. \tag{74b}$$

Sometimes it may be useful to rewrite (74a) by using

$$\begin{aligned} e_\alpha \rfloor\, (\vartheta^\gamma \wedge \Box\eta_\gamma) = -e_\alpha \rfloor^\star \{{}^\star\, (\vartheta^\gamma \wedge \Box\eta_\gamma)\} = -{}^\star\, ({}^\star\, (\vartheta^\gamma \wedge \Box\eta_\gamma) \wedge \vartheta_\alpha) \\ = -{}^\star\, (\vartheta^\gamma \wedge \Box\eta_\gamma) \wedge {}^\star\vartheta_\alpha = -{}^\star\, (\vartheta^\gamma \wedge \Box\eta_\gamma)\, \eta_\alpha \,, \end{aligned} \tag{75}$$

thereby finding

$$\left[\Box + \frac{1}{4}{}^\star \left(\vartheta^\beta \wedge \Box\eta_\beta\right)\right] \eta_\alpha = \ell^2\, \not\Sigma_\alpha \,, \tag{76}$$

or, alternatively:

$$\left[\Box - \frac{1}{4} \left(e_\beta \rfloor \Box\vartheta^\beta\right)\right] \eta_\alpha = \ell^2\, \not\Sigma_\alpha \ . \tag{77}$$

5.3 Yilmaz-Rosen and Schwarzschild solution compared

The Yilmaz-Rosen metric and the corresponding orthonormal coframe were displayed in (2) to (4). In order to compare the Yilmaz-Rosen metric with the Schwarzschild metric, we transform the former one from the isotropic coordinates used in (2) into Schwarzschild coordinates and Taylor expand it:

$$g_{00}^{\rm YR} = 1 - \frac{2m}{r} \qquad\qquad + \frac{25}{6}\frac{m^3}{r^3} + \mathcal{O}\left(\tfrac{m^4}{r^4}\right) \ , \tag{78a}$$

$$g_{11}^{\rm YR} = 1 + \frac{2m}{r} + \frac{5m^2}{r^2} + \frac{9m^3}{r^3} \ + \mathcal{O}\left(\tfrac{m^4}{r^4}\right) \ . \tag{78b}$$

For the Schwarzschild metric in Schwarzschild coordinates we find:

$$g_{00}^{\rm SS} = 1 - \frac{2m}{r} \qquad \text{(exact)} \ , \tag{79a}$$

$$g_{11}^{\rm SS} = 1 + \frac{2m}{r} + \frac{4m^2}{r^2} + \frac{8m^3}{r^3} \ + \mathcal{O}\left(\tfrac{m^4}{r^4}\right) \ . \tag{79b}$$

Their g_{00} components are equal up to second order. The radial components g_{11} begin to differ slightly in the second order. Therefore the Yilmaz-Rosen solution is consistent with the classical tests of general relativity and can, in particular, describe the post-Newtonian perihelion advance correctly, see Synge [24, page 296, footnote 1]. It requires further investigations to decide whether the two solutions can be observationally distinguished by strong gravity effects in close binary pulsar systems.

5.4 Yilmaz-Rosen metric motivated

A viable theory of gravitation should be consistent with the *local* equivalence principle. Let us consider an electromagnetic wave of frequency ω in the gravitational field of a point mass m. The frequency shift due to the propagation from a point with radial coordinate r to one with $r + \Delta r$ reads ($c = G = 1$):

$$\frac{\Delta\omega}{\omega} = \Delta U = -\frac{m\Delta r}{r^2} \ . \tag{80}$$

According to Mashhoon [25], Yilmaz effectively proposed to extend the local equivalence principle to a *global* one. With this idea in mind, we can tentatively integrate (80):

$$\int_{r_0}^{r_1} \frac{d\omega}{\omega} = \int_{r_0}^{r_1} dU \, . \tag{81}$$

We solve the integrals and find

$$\frac{\omega(r_1)}{\omega(r_0)} = \exp[U(r_1) - U(r_0)] \, . \tag{82}$$

Using the Newtonian potential explicitly, $U = -\frac{m}{r}$, and taking the limit $r_1 \to \infty$, yields

$$\frac{\omega(\infty)}{\omega(r)} = \exp\left(\frac{m}{r}\right) \, , \quad \text{and thus} \qquad \frac{\Delta t(\infty)}{\Delta t(r)} = \exp\left(-\frac{m}{r}\right) \, . \tag{83}$$

Since $\Delta t(\infty)$ is not influenced by gravity, one can directly read off

$$\vartheta^{\hat{t}} = e^{-\frac{m}{r}} \, dt \quad \text{or} \qquad g_{00} = e^{-\frac{2m}{r}} \, . \tag{84}$$

Following the pattern of the components of the Schwarzschild metric, we now define g_{11} as the inverse of g_{00}:

$$\widetilde{g} = e^{-\frac{2m}{r}} \, dt^2 - e^{\frac{2m}{r}} \, dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \, . \tag{85}$$

However, the Taylor approximation of this metric (which is *not* the Yilmaz-Rosen metric of (2)) reads:

$$\widetilde{g}_{00} = 1 - \frac{2m}{r} + \frac{2m^2}{r^2} - \frac{4}{3}\frac{m^3}{r^3} + \mathcal{O}\left(\frac{m^4}{r^4}\right) \, , \tag{86a}$$

$$\widetilde{g}_{11} = 1 + \frac{2m}{r} + \frac{2m^2}{r^2} + \frac{4}{3}\frac{m^3}{r^3} + \mathcal{O}\left(\frac{m^4}{r^4}\right) \, . \tag{86b}$$

The $\widetilde{g}_{00}$ component differs from g_{00}^{SS} already to second order. And the deviation of $\widetilde{g}_{11}$ from g_{11}^{SS} is doubled in comparison to that of the Yilmaz-Rosen g_{11}^{YR}. Therefore, in order to approximate the experimentally well verified Schwarzschild metric in an optimal way, we choose the forefactors of (85) as metric components in *isotropic* coordinates, which eventually leads to the Yilmaz-Rosen metric:

$$g^{\mathrm{YR}} = e^{-\frac{2m}{r}} \, dt^2 - e^{\frac{2m}{r}} \, \left(dx^2 + dy^2 + dz^2 \right) \, . \tag{87}$$

5.5 Yilmaz-Rosen solution and the vacuum field equation

The Yilmaz-Rosen metric (2), keeping in mind (3) and (4), fulfills the tracefree field equation (74a) [or, alternatively, (76) or (77)] with vanishing source. For a proof compare the corresponding computer algebra program in Sec. B. As we saw, the Yilmaz-Rosen solution is consistent with the classical tests of general relativity. However, its integration constant m cannot be directly identified with the source of a spherical body.

For that reason, we take recourse to the trace equation (74b). In the vacuum case we find

$$\lambda(x) = \frac{1}{4}\,{}^\star\left(\vartheta^\gamma \wedge \Box\eta_\gamma\right)\,. \tag{88}$$

The right hand side of this equation can be easily calculated with our computer algebra program, see also [2]. Therefore the Yilmaz-Rosen metric solves the vacuum field equation (22) provided the 'cosmological' function is prescribed as follows:

$$\lambda(x) = -\left(\frac{m}{r^2}e^{-\frac{m}{r}}\right)^2\,. \tag{89}$$

Since such an ad hoc structure looks too implausible to us, we can change horses at this moment: We reject the 'cosmological' function of Kaniel-Itin and put it to zero. Then we can mimic this function $\lambda(x)$ by means of the energy-momentum trace $-\ell^2\, T^\gamma{}_\gamma/4$, cf. (74b) and (71):

$$T^\gamma{}_\gamma = \left(\frac{2m}{\ell r^2}e^{-\frac{m}{r}}\right)^2\,. \tag{90}$$

This energy-momentum trace is plotted in Fig. 1. Therefore, the Yilmaz-Rosen solution fulfills the field equations

$$\Box\eta_\alpha - \frac{1}{4}e_\alpha\rfloor\left(\vartheta^\gamma \wedge \Box\eta_\gamma\right) = 0 \qquad \text{and} \qquad {}^\star\left(\vartheta^\gamma \wedge \Box\eta_\gamma\right) = -\left(\frac{2m}{\ell r^2}e^{-\frac{m}{r}}\right)^2\,. \tag{91}$$

Taking an ideal fluid for the description of matter, then, for vanishing pressure, $p=0$, we have $T^\gamma{}_\gamma = \rho$. Therefore we can understand the above computation as a matter distribution (90) which can be viewed as (probably unphysical) star model. The matter of such a star reaches to infinity, but it decreases exponentially. We find the maximum of the distribution at

$r = {}^{m}\!/_{2}$, compare with Fig. 1. Most of the star mass is concentrated inside the Schwarzschild radius $r_s = 2m$. The volume integral over $T^\gamma{}_\gamma$ yields the total mass m of the star:

$$\int T^\gamma{}_\gamma \, dV = 4\pi \int_0^\infty T^\gamma{}_\gamma \, r^2 \, dr = \int_0^\infty \frac{16\pi m^2}{\ell^2 r^2} \, e^{-\frac{2m}{r}} \, dr$$
$$\overset{x:=-\frac{2m}{r},\; dr=\frac{2m}{x^2}\,dx}{=} \quad \frac{8\pi m}{\ell^2} \int_{-\infty}^{0} e^x \, dx = \frac{8\pi m}{\ell^2} = M \;, \qquad (92)$$

where M is the mass in conventional units.

As a result, we can interpret the constant m of the Yilmaz-Rosen metric as the mass of a star, but this mass is distributed in a probably unphysical way.

6 Conclusion

We posed four questions about the Kaniel-Itin model in Sec. 1. We will try to answer them in turn:

(i) We can put the energy-momentum 3-form of matter on the right hand side of the KI-field equation (1), see (22):

$$[\Box + \lambda(x)] \, \eta_\alpha = \ell^2 \, \frac{\delta L_{\mathrm{mat}}}{\delta \vartheta^\alpha} =: \ell^2 \, \Sigma_\alpha \,. \qquad (93)$$

(ii) The Yilmaz-Rosen metric can be accommodated to (93) in the following sense: We decompose (93) in vacuum into

$$-\Box \eta_\alpha + \frac{1}{4} e_\alpha \rfloor \left(\vartheta^\beta \wedge \Box \eta_\beta \right) = 0 \;, \qquad (94a)$$

$$\frac{1}{4} {}^\star \left(\vartheta^\beta \wedge \Box \eta_\beta \right) = \lambda(x) \;. \qquad (94b)$$

These equations are fulfilled by the Yilmaz-Rosen metric, provided we prescribe the 'cosmological' function $\lambda(x)$ in the following way:

$$\lambda(x) = \frac{1}{4} {}^\star \left(\vartheta^\beta \wedge \Box \eta_\beta \right) = -\left(\frac{m}{r^2} e^{-\frac{m}{r}} \right)^2 \;. \qquad (95)$$

If one used a wave equation as field equation, i.e., if in (93) one put $\lambda(x) = 0$, then one could find the Yilmaz-Rosen solution for the matter distribution $T^\gamma{}_\gamma = \left(\frac{2m}{\ell r^2} e^{-\frac{m}{r}} \right)^2$.

(iii) There doesn't exist a consistent variational principle for arriving at (93). Maybe one is able to find one for the tracefree vacuum equation (94a).

(iv) The constraints of Kaniel-Itin on the variations amount to getting rid of the independence of the variations of the metric, provided the variations of the coframe are prescribed.

Is the model of Kaniel & Itin viable? Well, it is presently in intensive care ... And it is a beautiful model anyways.

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A Hodge star and η-basis

A.1 Elementary relations for the star etc.

We now collect some rules for calculations with the Hodge star, where ψ and ϕ are forms of the *same* degree p (see [12]):

$$^{\star\star}\psi = (-1)^{p(n-p)+\mathrm{ind}(g)}\,\psi\,, \tag{96}$$

$$^{\star}\psi \wedge \phi = {}^{\star}\phi \wedge \psi\,. \tag{97}$$

The *index* ind(g) of a metric is the number of minus signs if it is in diagonal form. Furthermore one has the useful rules

$$e_\alpha \rfloor {}^\star\psi = {}^\star\left(\psi \wedge \vartheta_\alpha\right)\,. \tag{98a}$$

$$e_\alpha \rfloor \psi = (-1)^{\mathrm{ind}(g)}\,{}^\star(\vartheta_\alpha \wedge {}^\star\psi)\,, \tag{98b}$$

$$^\star\left(e_\alpha \rfloor \psi\right) = (-1)^{p-1}\,\vartheta_\alpha \wedge {}^\star\psi\,, \tag{98c}$$

$$^\star\left(e_\alpha \rfloor {}^\star\psi\right) = (-1)^{(p+1)+\mathrm{ind}(g)}\,\psi \wedge \vartheta_\alpha\,. \tag{98d}$$

Sometimes we also need the formula:

$$\vartheta^\mu \wedge (e_\mu \rfloor \psi) = p\,\psi\,. \tag{99}$$

With these rules one can determine the η-basis, cf. (29):

$$\eta := {}^\star 1 = \frac{1}{n!}\eta_{\alpha_1\dots\alpha_n}\vartheta^{\alpha_1}\wedge\dots\wedge\vartheta^{\alpha_n} = \frac{1}{n!}\sqrt{|\det g_{\mu\nu}|}\,\vartheta^{\alpha_1}\wedge\dots\wedge\vartheta^{\alpha_n}\;, \tag{100a}$$

$$\begin{aligned}\eta^{\alpha_1\dots\alpha_p} &:= {}^\star\left(\vartheta^{\alpha_1}\wedge\dots\wedge\vartheta^{\alpha_p}\right) = \frac{1}{(n-p)!}\eta^{\alpha_1\dots\alpha_p}{}_{\alpha_{p+1}\dots\alpha_n}\vartheta^{\alpha_{p+1}}\wedge\dots\wedge\vartheta^{\alpha_n}\\ &= \frac{\sqrt{|\det g_{\mu\nu}|}}{(n-p)!}\,g^{\alpha_1\beta_1}\cdots g^{\alpha_p\beta_p}\,\epsilon_{\beta_1\cdots\beta_p\alpha_{p+1}\cdots\alpha_n}\vartheta^{\alpha_{p+1}}\wedge\dots\wedge\vartheta^{\alpha_n}\;,\end{aligned} \tag{100b}$$

$$\begin{aligned}\eta^{\alpha_1\dots\alpha_n} &:= {}^\star(\vartheta^{\alpha_1}\wedge\dots\wedge\vartheta^{\alpha_n})\\ &= \sqrt{|\det g_{\mu\nu}|}\,g^{\alpha_1\beta_1}\cdots g^{\alpha_n\beta_n}\,\epsilon_{\beta_1\cdots\beta_n} = \frac{1}{\sqrt{|\det g_{\mu\nu}|}}\,\epsilon^{\alpha_1\cdots\alpha_n}\;.\end{aligned} \tag{100c}$$

Two helpful rules, which connect the different elements of the η-basis, read

$$\eta^{\alpha_1\dots\alpha_p}{}_\mu = e_\mu\rfloor\eta^{\alpha_1\dots\alpha_p}\;, \tag{101a}$$

$$\vartheta^\mu\wedge\eta^{\alpha_1\dots\alpha_p} = \sum_{i=1}^{p}(-1)^{p-i}g^{\mu\alpha_i}\eta^{\alpha_1\dots\alpha_{i-1}\alpha_{i+1}\dots\alpha_p}\;. \tag{101b}$$

In case of independent variations of the metric components $g_{\alpha\beta}$, we need the rules

$$\delta g^{\alpha\beta} = -g^{\alpha\gamma}g^{\delta\beta}\,\delta g_{\gamma\delta}\;, \tag{102}$$

$$\delta\left[\det\left(g_{\mu\nu}\right)\right] = \det\left(g_{\mu\nu}\right)g^{\alpha\beta}\,\delta g_{\alpha\beta}\;. \tag{103}$$

A.2 Variation of the η-basis

With these definition and rules, we can compute a general variation of the η-basis, involving the fields ϑ^α and $g_{\alpha\beta}$:

$$
\begin{aligned}
\delta\eta^{\beta_1\cdots\beta_p} &= \frac{1}{(n-p)!}\,\delta\left(\eta^{\beta_1\cdots\beta_p}{}_{\beta_{p+1}\cdots\beta_n}\vartheta^{\beta_{p+1}}\wedge\cdots\wedge\vartheta^{\beta_n}\right)\\
&\overset{(100\mathrm{b})}{=} \frac{1}{(n-p)!}\,\eta^{\beta_1\cdots\beta_p}{}_{\beta_{p+1}\cdots\beta_n}\delta\left(\vartheta^{\beta_{p+1}}\wedge\cdots\wedge\vartheta^{\beta_n}\right)+\frac{1}{(n-p)!}\\
&\quad\times\delta\left(\sqrt{|\det g_{\mu\nu}|}\,g^{\alpha_1\beta_1}\cdots g^{\alpha_p\beta_p}\,\epsilon_{\alpha_1\cdots\alpha_p\beta_{p+1}\cdots\beta_n}\right)\vartheta^{\beta_{p+1}}\wedge\cdots\wedge\vartheta^{\beta_n}\\
&\overset{\substack{(26),(103),\\(102),(100\mathrm{b})}}{=} \frac{n-p}{(n-p)!}\,\eta^{\beta_1\cdots\beta_p}{}_{\beta_{p+1}\beta_{p+2}\cdots\beta_n}\left(\delta\vartheta^{\beta_{p+1}}\right)\wedge\left(\vartheta^{\beta_{p+2}}\wedge\cdots\wedge\vartheta^{\beta_n}\right)\\
&\qquad+\left(\frac{1}{2}g^{\kappa\lambda}\eta^{\beta_1\cdots\beta_p}-\sum_{i=1}^{p}g^{\kappa\beta_i}\eta^{\beta_1\cdots\beta_{i-1}\lambda\beta_{i+1}\cdots\beta_p}\right)\delta g_{\kappa\lambda}\\
&\overset{(100\mathrm{b})}{=} \delta\vartheta^{\beta_{p+1}}\wedge\eta^{\beta_1\cdots\beta_p}{}_{\beta_{p+1}}+\left(\frac{1}{2}\,g^{\kappa\lambda}\eta^{\beta_1\cdots\beta_p}\right.\\
&\qquad\left.-\sum_{i=1}^{p}(-1)^{p-i}g^{\kappa\beta_i}g^{\lambda\rho}e_\rho\rfloor\eta^{\beta_1\cdots\beta_{i-1}\beta_{i+1}\cdots\beta_p}\right)\delta g_{\kappa\lambda}\\
&\overset{(101\mathrm{b})}{=} \delta\vartheta^{\beta_{p+1}}\wedge\eta^{\beta_1\cdots\beta_p}{}_{\beta_{p+1}}\\
&\qquad+\left(\frac{1}{2}g^{\kappa\lambda}\eta^{\beta_1\cdots\beta_p}-g^{\lambda\rho}e_\rho\rfloor\left(\vartheta^\kappa\wedge\eta^{\beta_1\cdots\beta_p}\right)\right)\delta g_{\kappa\lambda}\,.
\end{aligned}
\tag{104}
$$

Hence

$$
\delta\eta^{\beta_1\cdots\beta_p}=\delta\vartheta^\mu\wedge\left(e_\mu\rfloor\eta^{\beta_1\cdots\beta_p}\right)+\left(\vartheta^{(\kappa|}\wedge\eta^{\beta_1\cdots\beta_p|\lambda)}-\frac{1}{2}\,g^{\kappa\lambda}\eta^{\beta_1\cdots\beta_p}\right)\delta g_{\kappa\lambda}\,.
\tag{105}
$$

A.3 Deduction of (33)

We start with (31). We abbreviate $\psi_p := \vartheta^{\beta_1} \wedge \cdots \wedge \vartheta^{\beta_p}$ and get, using the variation (105) of $\eta^{\beta_1\cdots\beta_p}$:

$$
\begin{aligned}
\psi_p \wedge \delta^\star\phi = {} & \delta\phi \wedge \eta^{\beta_1\cdots\beta_p} - \delta\psi_p \wedge {}^\star\phi + \phi \wedge \delta\vartheta^\mu \wedge \eta^{\beta_1\cdots\beta_p}{}_\mu \\
& + \phi \wedge \left(\vartheta^\kappa \wedge \eta^{\beta_1\cdots\beta_p\lambda} - \frac{1}{2}\, g^{\kappa\lambda}\eta^{\beta_1\cdots\beta_p}\right) \delta g_{\kappa\lambda} \\
\overset{(100b),\,(97)}{=} {} & \psi_p \wedge {}^\star(\delta\phi) - \delta\psi_p \wedge {}^\star\phi + \phi \wedge \delta\vartheta^\mu \wedge \left(e_\mu \rfloor \eta^{\beta_1\cdots\beta_p}\right) \\
& + \left(\phi \wedge \vartheta^\kappa \wedge \left(g^{\lambda\rho} e_\rho \rfloor \eta^{\beta_1\cdots\beta_p}\right) - \frac{1}{2}\,\psi_p \wedge g^{\kappa\lambda}\, {}^\star\phi\right) \delta g_{\kappa\lambda} \\
= {} & \psi_p \wedge \left({}^\star(\delta\phi) - \frac{1}{2}\,{}^\star\phi\; g^{\kappa\lambda}\,\delta g_{\kappa\lambda}\right) - \delta\vartheta^\mu \wedge (e_\mu \rfloor \psi_p) \wedge {}^\star\phi \\
& + (-1)^p \left(\delta\vartheta^\mu \wedge \phi \wedge (e_\mu \rfloor {}^\star\psi_p) + g^{\lambda\rho} e_\rho \rfloor (\phi \wedge \vartheta^\kappa) \wedge {}^\star\psi_p\; \delta g_{\kappa\lambda}\right) \\
= {} & \psi_p \wedge \left({}^\star(\delta\phi) - \frac{1}{2}\,{}^\star\phi\; g^{\kappa\lambda}\,\delta g_{\kappa\lambda}\right) + \delta\vartheta^\mu \wedge \Big(e_\mu \rfloor (\phi \wedge {}^\star\psi_p) \\
& - (e_\mu \rfloor \phi) \wedge {}^\star\psi_p - e_\mu \rfloor (\psi_p \wedge {}^\star\phi) + (-1)^p \psi_p \wedge (e_\mu \rfloor {}^\star\phi)\Big) \\
& + \psi_p \wedge \vartheta^\lambda \wedge {}^\star(\phi \wedge \vartheta^\kappa)\; \delta g_{\kappa\lambda} \\
\overset{(97)}{=} {} & \psi_p \wedge {}^\star(\delta\phi) - \psi_p \wedge {}^\star\left(\delta\vartheta^\mu \wedge (e_\mu \rfloor \phi)\right) + \psi_p \wedge \delta\vartheta^\mu \wedge (e_\mu \rfloor {}^\star\phi) \\
& - \psi_p \wedge \frac{1}{2}\,{}^\star\phi\; g^{\kappa\lambda}\,\delta g_{\kappa\lambda} + \psi_p \wedge \vartheta^\lambda \wedge g^{\kappa\rho} e_\rho \rfloor ({}^\star\phi)\; \delta g_{\kappa\lambda}\,.
\end{aligned}
\tag{106}
$$

Since $\psi_p := \vartheta^{\beta_1} \wedge \cdots \wedge \vartheta^{\beta_p}$ is constructed with p arbitrary ϑ^α's, we conclude for an arbitrary p-form ϕ:

$$
\begin{aligned}
(\delta^\star - {}^\star\delta)\,\phi = {} & \delta\vartheta^\alpha \wedge (e_\alpha \rfloor {}^\star\phi) - {}^\star\left[\delta\vartheta^\alpha \wedge (e_\alpha \rfloor \phi)\right] \\
& + \delta g_{\alpha\beta}\left[\vartheta^{(\alpha} \wedge (e^{\beta)} \rfloor {}^\star\phi) - \frac{1}{2}\, g^{\alpha\beta}\, {}^\star\phi\right].
\end{aligned}
\tag{107}
$$

For the special choice of an *orthonormal* coframe, we have $\delta g_{\alpha\beta} = 0$. In this case the two last summands vanish:

$$
(\delta^\star - {}^\star\delta)\,\phi = \delta\vartheta^\alpha \wedge (e_\alpha \rfloor {}^\star\phi) \;-\; {}^\star\left[\delta\vartheta^\alpha \wedge (e_\alpha \rfloor \phi)\right]. \tag{108}
$$

B Computer algebra program

The following `Reduce` program was written with the help of the `Excalc` package, see [26, 27].[5] It verifies (i) the decompositon (58) of the KI-Lagrangian, (ii) that the Yilmaz-Rosen metric fulfills the tracefree KI-vacuum field equation (74a) and (77), and (iii) the validity of Eq. (89).

```
% file kaniti.exi, 1998-01-11, fwh+fg     %in "kaniti.exi";

load_package excalc$

%
% Basic definitions:
%

pform psi=0, r=0, lam=0$
fdomain psi=psi(x,y,z), r=r(x,y,z), lam=lam(x,y,z)$

coframe o(t) =  psi    * d t  + a*sin(x)* d x,
        o(x) = (1/psi) * d x  + b*sinh(z)*d y,
        o(y) = (1/psi) * d y,
        o(z) = (1/psi) * d z  with signature(+1,-1,-1,-1)$

frame e$
sgn :=-1$

%
% Checking the decompositon (58) of the KI-Lagrangian
%

pform v0rumpf4=4, v1rumpf4=4, v2rumpf4=4, v3rumpf4=4,
      v4rumpf4=4, vki4=4$

v0rumpf4 :=    o(a)  ^ #   o(-a)/4$
```

[5]This program works properly only with a new patch of `Excalc`, fixing an earlier bug on the hodge dual of scalars. Older versions of `Excalc` need as additional input the following function which should be put in after `Excalc` has been loaded:

```
symbolic procedure dual0 u;
(multpfsq(mkwedge ('wedge .  basisforml!*),
simpexpt list(mk!*sq(absf!* numr x ./
absf!* denr x),'(quotient 1 2))))
where x = simp!* detm!*;
```

```
v1rumpf4 :=  d o(a)  ^ # d o(-a)$
v2rumpf4 := (d o(-a) ^     o(a)) ^ # (d o(-b) ^ o(b) )$
v3rumpf4 := (d o(a)  ^     o(b)) ^ # (d o(-a) ^ o(-b))$
v4rumpf4 := (d o(-a) ^     o(b)) ^ # (d o(-b) ^ o(a) )$

vki4 := rho0 * v0rumpf4 + rho1 * v1rumpf4 + rho2 * v2rumpf4
                        + rho3 * v3rumpf4 + rho4 * v4rumpf4;

rho0 := 0; rho1 := 2; rho2 := 0; rho3 := 0; rho4 :=-1;

diff := vki4 -  (  d    o(a)   ^ #(   d   o(-a))
             +   #(d (# o(a))) ^ #(# (d # o(-a))) );

diff := diff;

% diff has to vanish. Note that our check is not for
% a general coframe. In the following we choose the
% Yilmaz-Rosen coframe and set a=0, b=0:

a:=0$
b:=0$

r**2 := (x**2+y**2+z**2)$
@(r,x):= x/r; @(r,y):= y/r; @(r,z):= z/r$
psi := exp(-m/r)$

pform dalembertcof1(a)=1, dalemberteta3(a)=3,
kifeqtrfreea3(a)=3, kifeqtrfreeb3(a)=3$

dalembertcof1(a):= -d(#(d(#   o(a) ))) - #(d(#(d  o(a) )));
dalemberteta3(a):= -d(#(d(#(# o(a))))) - #(d(#(d(#o(a)))));

%
% Checking the vacuum field equation (1) of Kaniel & Itin
%

% lhs of Eq.(74a)
kifeqtrfreea3(a) := dalemberteta3(a) - e(a) _|
                    ( o(b) ^ dalemberteta3(-b) )/4;

% lhs of Eq.(77)
```

```
kifeqtrfreeb3(a) := dalemberteta3(a) - ( e(-b) _|
                    (dalembertcof1(b))/4 )  ^  # o(a);

% Eq.(88)
lam  := #(o(b) ^ dalemberteta3(-b))/4;

% Eq.(89)
lam - ( - ((m/r**2)*e**(-m/r))**2);

% Eq.(90)
energytrace := - 4 * lam / ell**2;

end$
```

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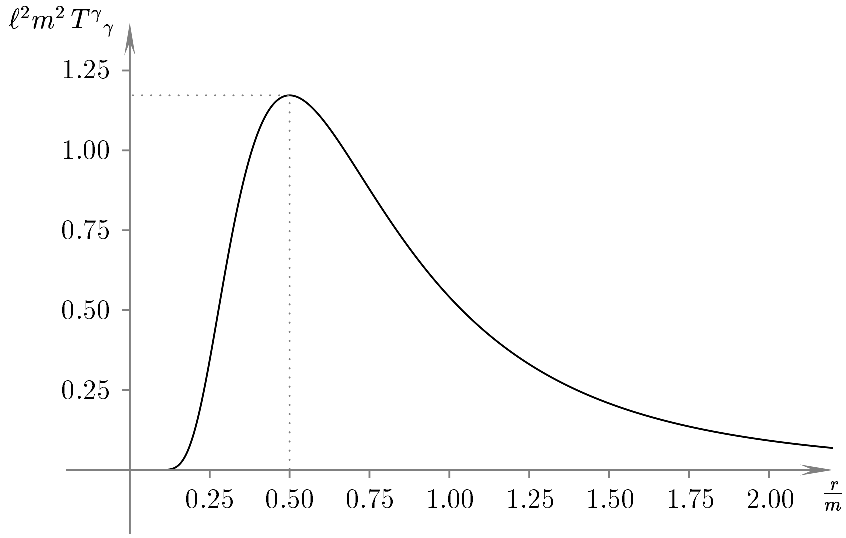

Figure 1: *The prescribed matter distribution* $\ell^2\, T^\gamma{}_\gamma = \left(\frac{2m}{r^2} e^{-\frac{m}{r}}\right)^2$. *Most of the matter is inside the Schwarzschild radius* $r_s = 2m$.